D 114585 **(Pages : 4)** **Name..................................**

Reg. No...............................

FIRST SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2024

(CBCSS)

Mathematics

MTH 1C 03—REAL ANALYSIS—I

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A (Short Answer Type Questions)

Answer **all** *questions.*

Each question carries a weightage 1.

1. Prove that balls are convex.
2. Discuss the continuity/discontinuity behaviour of the function f defined by :

$$f(x) = \begin{cases} x & (x \text{ rational}) \\ 0 & (x \text{ irrational}). \end{cases}$$

3. Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then prove that f is continuous at x.
4. Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$.

   Then prove that fg is differentiable at x.
5. Let f be defined on $[a, b]$. If f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then prove that $f'(x) = 0$.
6. Define Riemann integrable function. Give an example.
7. Describe briefly : Compare pointwise convergence and uniform convergence of sequence of functions.
8. Define uniformly bounded sequence of functions.

(8 × 1 = 8 weightage)

Turn over

Part B (Paragraph Type Questions)

Answer any **two** *questions from each module.*

Each question carries a weightage 2.

Module I

9. Prove that every infinite subset of a countable set A is countable.

10. Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that

    $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \ldots)$, then prove that $\bigcap_1^\infty I_n$ is not empty.

11. Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and h is the mapping of E into Z defined by :

    $$h(x) = g(f(x)) \qquad (x \in E).$$

    If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then prove that h is continuous at p.

Module II

12. Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f and g is differentiable at the point $f(x)$.

    If

    $$h(t) = g(f(t)) \qquad (a \le t \le b),$$

    then prove that h is differentiable at x, and

    $$h'(x) = g'(f(x))\, f'(x).$$

13. Prove that $\int_{-a}^{a} f\, d\alpha \le \int_{a}^{b} f\, d\alpha.$

14. If $f \in \Re$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then prove that :

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Module III

15. If γ' is continuous on $[a, b]$, then prove that γ is rectifiable, and :

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

16. Give an example to show that limit of the integral need not be equal to the integral of the limit, even if both are finite.

17. For every interval $[-a, a]$ prove that there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that

$$\lim_{x \to \infty} P_n(x) = |x|$$

uniformly on $[-a, a]$.

(6 × 2 = 12 weightage)

Part C (Essay Type Questions)

Answer any **two** *questions.*
Each question carries a weightage 5.

18. Let E be a subset of a metric space X Prove that E is open if and only if its complement is closed.

19. Prove that $f \in \Re(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

20. If **f** maps $[a, b]$ into $\mathbb{R}^k$ and if $f \in \Re(\alpha)$ for some monotonically increasing function α on $[a, b]$, then prove that If $|\mathbf{f}| \in \Re(\alpha)$, and

$$\left|\int_a^b \mathbf{f}\, d\alpha\right| \le \int_a^b |\mathbf{f}|\, d\alpha.$$

Turn over

21. Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f'_n\}$ converges uniformly on $[a, b]$, then prove that $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and

$$f'(x) = \lim_{x \to \infty} f'_n(x) \qquad (a \le x \le b).$$

(2 × 5 = 10 weightage)