D 114587 **(Pages : 3)** **Name....................................**

Reg. No................................

FIRST SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2024

(CBCSS)

Mathematics

MTH 1C 05—NUMBER THEORY

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A (Short Answer Type Questions)

Answer **all** *questions.*

Each question carries a weightage 1.

1. Define multiplicative function and prove that Euler totient function $\phi(n)$ is multiplicative.

2. State and prove Generalized Mobius inversion formula.

3. Define the big oh notation.

4. For $x \geq 1$, show that $\sum_{n \leq x} \mu(n)\left[\frac{x}{n}\right] = 1$.

5. For $x \geq 2$, show that $\vartheta(x) = \pi(x)\log(x) - \int_x^2 \frac{\pi(t)}{t}\,dt$.

6. Explain enciphering and deciphering transformation.

7. Define Legendre's symbol and evaluate the Legendre symbol $(5|p)$,where p is an odd prime.

8. State Gauss lemma.

(8 × 1 = 8 weightage)

Turn over

Part B (Paragraph Type Questions)

Answer any **two** *questions from each module.*

Each question carries a weightage 2.

MODULE I

9. If $n \geq 1$, show that $\phi(n) = \sum_{d/n} \mu(d) \frac{n}{d}$.

10. If f is an arithmetical function with $f(1) \neq 0$, show that there is a unique arithmetical function f^{-1} such that $f * f^{-1} = f^{-1} * f = 1$.

11. For $x \geq 1$, show that $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{t-s}}{1-s} + \varsigma(s) + O\left(x^{-3}\right)$, if $s > 0, s \neq 1$.

MODULE II

12. Show that for $x > 0, 0 \leq \frac{\Psi(x)}{x} - \frac{\vartheta(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x}\log 2}$.

13. State and prove Abel's identity.

14. For all $x \geq 1$, show that $\sum_{n \leq x} \frac{\Lambda(n)}{n} = x \log x + O(1)$.

MODULE III

15. Determine whether 888 is a quadratic residue or non residue mod 1999.

16. Evaluate Legendre's symbol $(1|p)$ and $(2|p)$, where p is an odd prime.

17. State and prove Quadratic Reciprocity Law for Jacobi's symbol.

(6 × 2 = 12 weightage)

Part C (Essay Type Questions)

Answer any **two** *questions.*

Each question carries a weightage 5.

18. Let f be multiplicative. Prove that if f is multiplicative if and only if

    $f^{-1}(n) = \mu(n) f(n)$ for all $n \geq 1$.

19. State and prove Euler's summation formula.

20. Let p_n denote the n^{th} prime. Show that the following relations are logically equivalent.

    (a) $\lim_{x \to \infty} \dfrac{\pi(x) \log(x)}{x} = 1.$

    (b) $\lim_{x \to \infty} \dfrac{\pi(x) \log \pi(x)}{x} = 1.$

    (c) $\lim_{x \to \infty} \dfrac{P_n}{n \log n} = 1.$

21. Prove that The Diophantine equation $y^2 = x^3 + k$ has no solutions if k has the form

    $k = (4n-1)^3 - 4m^2$, where m and n are integers such that no prime $p \equiv -1 \pmod 4$ divides m.

(2 × 5 = 10 weightage)