D 122512 **(Pages : 3)** **Name.....................................**

Reg. No.................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2025

(CBCSS)

Mathematics

MTH2C06—ALGEBRA—II

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*

Each question carries weightage 1.

1. Give a maximal ideal of the ring $\mathbb{Z}$ of integers.
2. Find a prime ideal of the ring $\mathbb{Z}_6$ of integers mod 6.
3. Verify whether $\sqrt{1+\sqrt{2}}$ is algebraic over the field $\mathbb{Q}$ of rationals.
4. Does there exist a field of 8 elements. Justify your answer.
5. Find a nonidentity automorphisms of the field $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$.
6. Find the splitting field of the polynomial $x^4 - 5x^2 + 6$ over $\mathbb{Q}$.
7. Let K = $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ and E = $\mathbb{Q}\left(\sqrt{3}\right)$. Find the subgroup of G (K/F) that correspond to E in the Galois correspondence.
8. Verify whether the polynomial $x^5 - 1$ is solvable by radicals over $\mathbb{Q}$.

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any **two** *questions from each module.*

Each question carries weightage 2.

Module I

9. Let R be a ring with unity and of characteristic zero. Show that R contains a subring isomorphic to the ring $\mathbb{Z}$ of integers.

10. Consider the ring $\mathbb{Z} \times \mathbb{Z}$. Show that $\mathbb{Z} \times \{0\}$ is a prime ideal which is not a maximal ideal.

11. Show that $\mathbb{Q}\left(\sqrt{2}\right)$ is an algebraic extension of $\mathbb{Q}$.

Module II

12. Show that every finite extension of a finite field is a simple extension.

13. Describe all automorphisms of the splitting field K of $x^3 - 1 \in \mathbb{Q}[x]$.

14. Show that if $p(x)$ is irreducible in F $[x]$ then all zeros of $p(x)$ in $\overline{\mathrm{F}}$ have the same multiplicity.

Module III

15. Let $\mathrm{K} = \mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ and $\mathrm{G} = \mathrm{G}(\mathrm{K}/\mathbb{Q})$ be the Galois group. Find the subfield E of K corresponding to the subgroup :

$$\mathrm{H} = \left\{\sigma \in \mathrm{G} : \sigma\left(\sqrt{2}\right) = \left(\sqrt{2}\right)\right\}.$$

of G in the Galois correspondence.

16. Show that the 4^{th} cyclotomic polynomial $\phi_4(x)$ is $x^2 + 1$.

17. Show that the Galois group of the p^{th} cyclotomic extension of $\mathbb{Q}$ for a prime p is the cyclic group of order $p - 1$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*

Each question carries weightage 5.

18. (a) Let F be a field. Show that every ideal in F $[x]$ is a principal ideal.

(b) Show that if $p(x)$ is irreducible in F $[x]$ then the ideal $\langle p(x) \rangle$ is a maximal ideal in F (x).

19. (a) Show that every finite extension of a finite field F is an algebraic extension of F.

    (b) Find the irreducible polynomial for $\sqrt[3]{1+\sqrt{2}}$ over the field Q of rationals.

20. (a) Define splitting field.

    (b) Let $F \leq E \leq \overline{F}$ be fields and let E be a splitting field over F. Show that every automorphism of $\overline{F}$ leaving F fixed induces an automorphism of E.

    (c) Let $p(x)$ be an irreducible polynomial in $F[x]$ and E be the splitting field of $p(x)$. Let σ be an automorphism of E leaving F fixed. Show that if $\alpha \in E$ is a zero of $p(x)$ then $\sigma(\alpha)$ is also a zero of $p(x)$.

21. (a) Define Galois group of an extension.

    (b) Let K be a finite normal extension of a field F. Let λ be the correspondence between intermediate fields of the extension and subgroups of the Galois group G(K/F). Prove that :

    (i) λ is one to one and onto.

    (ii) if E_1, E_2 are intermediate fields such that $E_1 \leq E_2$ then $\lambda E_2 \leq \lambda E_1$.

(2 × 5 = 10 weightage)