D 122513 **(Pages : 4)** **Name.......................................**

Reg. No...................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2025

(CBCSS)

Mathematics

MTH 2C 07—REAL ANALYSIS—II

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Verify whether $\{1, 2, 3\}$ is a Borel set.

2. Let $E_n = \left(-\frac{1}{n}, 1+\frac{1}{n}\right)$. Find $m\left(\bigcap_n E_n\right)$ where m is the Lebesgue measure.

3. Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}.$$

Verify whether f is a Lebesgue measurable function on $\mathbb{R}$.

4. Let

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x \le 2 \end{cases}.$$

Verify whether $f(x)$ is a characteristic function.

Turn over

5. Let $f(x)$ in $[0,1]$ be defined by

$$f(x)=\begin{cases}0 \text{ if } x \text{ is rational} \\ 1 \text{ otherwise}\end{cases}.$$

Find $\int_E f$ where E is the set of all irrationals in $[0,1]$.

6. Consider the family of functions

$$F=\left\{\chi_{\left[-\frac{1}{n},\frac{1}{n}\right]} : n \in \mathbb{N}\right\}.$$

Verify whether F is uniformly integrable in $[-1,1]$.

7. Let $(q_n)_{n=1}^{\infty}$ be an enumeration of the rationals in $(0,1)$ and let

$$f(x)=\sum_{n : q_n \leq x} \frac{1}{2^n}$$

for $0<x<1$. Show that f is not continuous at q_k for any k.

8. Let $f(x)$ be continuous and increasing on $[0.1]$ Show that $f(x)$ is absolutely continuous on $[0,1]$.

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each module.*
Each question has weightage 2.

Module I

9. Let C be a countable subset of $\mathbb{R}$. Show that $m^*(C)=0$ where m^* is the Lebesgue outer measure on $\mathbb{R}$.

10. Show that if $m^*(E)=0$ then E is Lebesgue measurable.

11. Show that if (A_n) is an ascending chain of Lebesgue measurable sets then

$$m\left(\bigcup_n A_n = \lim_{n\to\infty} m(A_n)\right).$$

Module II

12. Let E be a measurable set of finite measure and ϕ, ψ be simple functions on E such that $\phi \leq \psi$. Show that

$$\int_E \phi \leq \int_E \psi.$$

13. Let E be a set of measure zero and f be a bounded measurable function on E. Show that $\int_E f = 0$.

14. Let (f_n) be a sequence of non negative integrable functions on a measurable set E and let $\lim_{n\to\infty} \int_E f_n = 0$. Show that (f_n) converges to 0 in measure on E.

Module III

15. Let f be a function on $[0,1]$ such that $f(x) = g(x) - h(x)$ for all $x \in [0,1]$ where g and h are increasing functions. Show that f is of bounded variation on [0,1].

16. Let f be absolutely continuous on $[0,1]$. Show that the total variation function $TV(f)$ is also absolutely continuous on $[0,1]$.

17. Let ϕ be a convex function on a bounded interval (a,b). Show that ϕ is a Lipschitz function on every subinterval $[c,d]$ of (a,b).

(6 × 2 = 12 weightage)

Turn over

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. (a) Define Lebesgue measurable set.

    (b) Show that union of a finite collection of measurable sets is measurable.

    (c) Show that the union of a countable collection of measurable sets is measurable.

19. Let f, g be measurable functions on a measurable set E. Show that

    (a) $f + g$ is measurable.

    (b) f^2 is measurable.

    (c) if $h(x) = f(x)\, g(x)$ for all $x \in \mathrm{E}$ then h is measurable.

20. (a) Define convergence in measure.

    (b) Let (f_n) be a sequence of functions defined on a measurable set E of finite measure. Suppose that $f_n \to f$ pointwise a.e. on E and that f is finite a.e. on E. Show that $f_n \to f$ in measure on E.

21. (a) Define indefinite integral of a function.

    (b) Show that a function f on a closed and bounded interval $[a, b]$ is an indefinite integral over $[a, b]$ if and only if f is absolutely continuous on $[a, b]$.

(2 × 5 = 10 weightage)