D 122515 **(Pages : 5)** **Name...**

Reg. No....................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2025

(CBCSS)

Mathematics

MTH 2C 09—ODE AND CALCULUS OF VARIATIONS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Section A (Short Answer Type Questions)

Answer **all** *questions.*
Each question carries a weightage 1.

1. Find the series solution of the differential equation $y' = xy$.

2. What it mean that $x = x_0$ is a regular singular point of the differential equation

$$y'' + P(x)\, y' + Q(x)\, y = 0.$$

3. Determine the nature of the point $x = 0$ for the differential equation

$$y'' + (\sin x)\, y = 0.$$

4. Prove that $\frac{d}{dx} J_0(x) = -J_1(x)$.

5. Find the critical points of system

$$\begin{cases} \frac{dx}{dt} = y^2 - 5x + 6 \\ \frac{dy}{dt} = x - y \end{cases}.$$

6. Is the function $x^2 - xy - y^2$ positive definite ? Justify your answer.

Turn over

7. Find the exact solution of the initital value problem

$$y' = 2x(1+y),\ y(0) = 1.$$

8. State a sufficient condition ensuring the local existence and uniqueness of a solution for an initial value problem

$$y' = f(x, y),\ y(x_0) = y_0.$$

(8 × 1 = 8 weightage)

Part B (Paragraph Type Questions)

Answer any **two** *questions from each module.*
Each question carries a weightage 2.

Module I

9. Find a series solution of the differential equation $\left(1+x^2\right)y'' + 2xy' - 2y = 0$.

10. Prove that the differential equation

$$4x^2y'' - 8x^2y' + \left(4x^2 + 1\right) = 0.$$

has only one Frobenius series solution. Also find the general solution.

11. Prove that

$$x\left[\lim_{a\to\infty} F\left(a, a, \frac{3}{2}, \frac{-x^2}{4a^2}\right)\right] = \sin x.$$

Module II

12. Find the general solution of the system

$$\begin{cases} \dfrac{dx}{dt} = -3x + 4y \\ \dfrac{dy}{dt} = -2x + 3y \end{cases}.$$

13. Describe the phase portrait of the system

$$\begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = 0 \end{cases}.$$

14. Show that $(0, 0)$ is an asymptotically stable critical point of

$$\begin{cases} \dfrac{dx}{dt} = -y - x^3 \\ \dfrac{dy}{dt} = x - y^3 \end{cases}$$

Module III

15. For what points (x_0, y_0) does the initial value problem

$$y' = y|y|, \;\; y(x_0) = y_0$$

have a unique solution on some interval $|x - x_0| \le h$.

16. Find the extremals for the integral

$$\int_{x_1}^{x_2} f(x, y, y')\, dx$$

if :

$$f(x, y, y') = y^2 - (y')^2.$$

17. Show that the zeros of the functions $a \sin x + b \cos x$ and $c \sin x + d \cos x$ are distinct and occur alternately whenever $ad - bc \neq 0$.

(6 × 2 = 12 weightage)

Turn over

Part C (Essay Type Questions)

Answer any **two** *from the following four questions (*18-21*).*

Each question carries a weightage 5.

18. Prove that

$$\int_{-1}^{1} \mathrm{P}_m(x)\,\mathrm{P}_n(d)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}.$$

19. (a) Prove that

$$\frac{d}{dx}\left[x^p\,\mathrm{J}_p(x)\right] = x^p\,\mathrm{J}_{p-1}(x).$$

(b) If $\mathrm{W}(t)$ is the Wronskian of the two solutions of the homogeneous system

$$\begin{cases} \dfrac{dx}{dt} = a_1(t)\,x + b_1(t)\,y \\ \dfrac{dy}{dt} = a_2(t)\,x + b_2(t)\,y \end{cases},$$

then prove that $\mathrm{W}(t)$ is identically zero or nowhere zero.

20. Consider the linear system

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y \\ \dfrac{dy}{dt} = a_2x + b_2y \end{cases}$$

with auxiliary equation

$$m^2 - (a_1 + b_2)\,m + (a_1b_2 - a_2ba_1) = 0.$$

If the roots m_1 and m_2 of the auxiliary equation are real and equal, then prove that the critical point $(0, 0)$ is a nod.

21. (a) Find the stationary function of

$$\int_0^4 \left[xy' - (y')^2 \right] dx$$

which is determined by the boundary conditions $y(0) = 0$ and $y(4) = 3$.

(b) Let $y(x)$ and $z(x)$ be non-trivial solutions of

$$y'' + q(x)\, y = 0$$

and

$$z'' + r(x)\, z = 0$$

where $q(x)$ and $r(x)$ are positive functions such that $q(x) > r(x)$ Prove that $y(x)$ vanishes at least once between any two successive zeros of $z(x)$.

(2 × 5 = 10 weightage)

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Mathematics

MTH 2C 09—ODE AND CALCULUS OF VARIATIONS

(2019 Admission onwards)

[Improvement Candidates need not appear for MCQ Part]

(Multiple Choice Questions for SDE Candidates)

Time : 20 Minutes	**Total No. of Questions : 20**	**Maximum : 5 Weightage**

INSTRUCTIONS TO THE CANDIDATE

1. This Question Paper carries Multiple Choice Questions from 1 to 20.
2. The candidate should check that the question paper supplied to him/her contains all the 20 questions in serial order.
3. Each question is provided with choices (A), (B), (C) and (D) having one correct answer. Choose the correct answer and enter it in the main answer-book.
4. The MCQ question paper will be supplied after the completion of the descriptive examination.

MTH 2C 09—ODE AND CALCULUS OF VARIATIONS

(Multiple Choice Questions for SDE Candidates)

1. Which of the following are two independent solutions of the differential equation $y'' + y = 0$?

(A) $\text{Cos } x, \text{Sin } x$.
(B) $\text{Cos } x, e^x$.
(C) $\text{Sin } x, e^x$.
(D) e^x, e^{-x}.

2. Which of the following forms a basis of the differential equation $4x^2y'' - 3y = 0$?

(A) x^2, x^3.
(B) $x^{-1/2}, x^{-3/2}$.
(C) $x^{-1/2}, x^{3/2}$.
(D) $e^x, \sin x$.

3. The radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is :

(A) 0.
(B) ∞.
(C) 1
(D) $1/n!$.

4. Solution of the differential equation $y' = y$ is :

(A) $y = \sin x$.
(B) $y = \cos x$.
(C) $y = \tan^{-1} x$.
(D) $y = e^x$.

5. The series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots$ is the expansion of the function :

(A) $\text{Sin } x$.
(B) $\text{Cos } x$.
(C) $\text{Tan}^{-1} x$.
(D) $\text{Sin}^{-1} x$.

6. Whichof the following points is not an ordinary point of the differential equation $x^2y'' + (\sin x)\, y = 0$?

(A) $x = 1$.
(B) $x = 2$.
(C) $x = 0$.
(D) $x = 4$.

7. If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$, then which of the following is always a solution :

(A) $y_1(x)\ y_2(x)$.

(B) $y_1(x)/y_2(x)$.

(C) $y_1^2(x) + y_2^2(x)$.

(D) $y_1(x) + y_2(x)$.

8. The general solution of the differential equation $y'' + y = 0$ is :

(A) $y = c_1 \cos x + c_2 \sin x$.

(B) $y = c_1 \cos 2x + c_2 \sin 2x$.

(C) $y = \cos x$.

(D) $y = c_1 e^x + c_2 x\, e^x$.

9. The Legendre's equation $\left(1 - x^2\right) y'' - 2xy' + p(p+1)y = 0$ has :

(A) A singularity at the origin.

(B) Origin as a regular singular point.

(C) $1, -1$ as singular points.

(D) None of the above.

10. The singular points of the Gauss's hyper geometric equation are :

(A) $x = 0$ and $x = 1$.

(B) $x = 0$ and $x = -1$.

(C) $x = 0$ and $x = 2$.

(D) $x = 1$ and $x = -1$.

11. If $m = n$, then $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx =$

(A) 0.

(B) 2.

(C) $\frac{2n+1}{2}$.

(D) $\frac{2}{2n+1}$.

12. If $P_n(x)$ is the n^{th} Legendre polynomial then $P_n(1) =$

(A) 1.

(B) $(-1)^n$.

(C) 0.

(D) ∞.

13. If $P_n(x)$ denotes the n^{th} Legendre polynomial, then $P_1(x) =$

(A) x.

(B) 0.

(C) 1.

(D) $(-1)^n$.

14. If $P_n(x)$ is the n^{th} Legendre polynomial then, $P_2(-1) =$

(A) 0.

(B) – 1.

(C) 1.

(D) 2.

15. The Bessel function of the first kind of order 'P' is given by $J_p(x) =$

(A) $\sum_{n=0}^{\infty} \frac{(-1)^n \left(x/2\right)^{2n+p}}{n!(p+n)!}$.

(B) $\sum_{n=0}^{\infty} \frac{\left(x/2\right)^{2n+p}}{n!(n+p)!}$.

(C) $\sum_{n=0}^{\infty} \frac{(-1)^{n+p} \left(x/2\right)^{n+p}}{n!(p+n)!}$.

(D) $\sum_{n=0}^{\infty} \frac{(-1)^n \left(x/2\right)^{2p+n}}{n!(n+p)!}$.

16. Which of the following system is an example for a non homogeneous system of equations ?

(A) $\frac{dx}{dt} = x, \frac{dy}{dt} = y$.

(B) $\frac{dx}{dt} = x\,t, \frac{dy}{dt} = y$.

(C) $\frac{dx}{dt} = x\,t, \frac{dy}{dt} = yt$.

(D) $\frac{dx}{dt} = x + 1, \frac{dy}{dt} = y$.

17. The only critical point of the system $\frac{dx}{dt} = x, \frac{dy}{dt} = -x + 2y$ is :

(A) $x = 1, y = 1$.

(B) $x = 1, y = -1$.

(C) Origin.

(D) $x = 0, y = 1$.

18. Suppose that $-1+i$ and 2 are the roots of the auxiliary equation of a linear system of differential equations and if origin is a critical point of the system, then it is :

(A) Stable. (B) Unstable.

(C) Asymptotically stable. (D) None of these.

19. Which of the following is true about the solutions of the equations $y'' + 4y = 0 - (1)$ and $y'' + y = 0 - (2)$?

(A) Zeros of the solutions of (1) and (2) are same.

(B) Zeros of solutions of (1) oscillate more rapidly that the zeros of solution of (2).

(C) Zeros of solutions of (1) and (2) are finite.

(D) None of the above.

20. First approximation to the solution of the initial value problem $y' = y^2, y(0) = 1$ is :

(A) $y_1 = 1$. (B) $y_1 = 1 + x$.

(C) $y_1 = 1 + x + x^2$. (D) $y_1 = 0$.