D 111186 **(Pages : 4)** **Name.....................................**

Reg. No...............................

THIRD SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2024

(CBCSS)

Mathematics

MTH3C11—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A (Short Answer Type Questions)

Answer **all** *questions.*

Each question carries a weightage 1.

1. Define linearly independent set of vectors. Illustrate with an example.

2. Give an example of a linear transformation between two vector spaces and find its derivative.

3. Let E be an open set in $\mathbb{R}^n$, and $f : E \to \mathbb{R}^m$. When we say that f is continuously differentiable in E.

4. Define parametrized curve in $\mathbb{R}^n$. Give a parametrization of the astroid $x^{2/3} + y^{2/3} = 1$.

5. Compute the curvature of the curve :

$$\gamma(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right).$$

6. Describe briefly : Stereographic projection from S^2 to the plane.

7. Let $\sigma(u,v)$ be a surface patch with standard unit normal $N(u, v)$. Then prove that $N_u \cdot \sigma_u = -L$.

8. Prove that the second fundamental form of a plane is zero.

(8 × 1 = 8 weightage)

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Part B (Paragraph Type Questions)

Answer any **two** *questions from each module.*
Each question carries a weightage 2.

Module I

9. Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then prove that dim $X \le r$.

10. If X is a complete metric space, and if φ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\varphi(x) = x$.

11. If $A \in L\left(\mathbb{R}^{n+m}, \mathbb{R}^n\right)$ and if A_x is invertible, then prove that there corresponds to every $k \in \mathbb{R}^m$ a unique $h \in \mathbb{R}^n$ such that $A(h, k) = 0$.

Module II

12. If the tangent vector of a parametrized curve is constant, then prove that the image of the curve is part of a straight line.

13. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ be a unit-speed curve, let $s_0 \in (\alpha, \beta)$ and let φ_0 be such that

$$\dot{\gamma}(s_0) = (\cos \varphi_0, \sin \varphi_0).$$

Then prove that there is a unique smooth function $\varphi : (\alpha, \beta) \to \mathbb{R}$ such that $\varphi(s_0) = \varphi_0$ and that the equation

$$\dot{\gamma}(s) = (\cos \varphi(s), \sin \varphi(s)).$$

holds for all $s \in (\alpha, \beta)$.

14. Let $f : S_1 \to S_2$ be a diffeomorphism. If σ_1 is an allowable surface patch on S_1, then prove that $f \circ \sigma_1$ is an allowable surface patch on S_2.

Module III

15. Prove that

$$\| \sigma_u \times \sigma_v \| = \left(EG - F^2\right)^{1/2}.$$

16. Obtain the relations connecting the curvature, normal curvature and geodesic curvature of a normal section of a surface.

17. Prove that the principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point. Also prove that the principal vectors are the tangent vectors of the curves giving these maximum and minimum values.

(6 × 2 = 12 weightage)

Part C (Essay Type Questions)

Answer **two** *questions.*

Each question carries a weightage 5.

18. (a) If $A \in L\left(\mathbb{R}^n, \mathbb{R}^m\right)$ and $B \in L\left(\mathbb{R}^m, \mathbb{R}^k\right)$, then prove that $\|BA\| \le \|B\| \, \|A\|$.

(b) Suppose f maps on an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and f is differentiable at a point $x \in E$. Then prove that the partial derivatives $(D_j f_i)(x)$ exist, and

$$f'(x)\, e_j \sum_{i=1}^{m} \left(D_j\, f_i\right)(x)\, u_i \qquad (1 \le j \le n),$$

where $\{e_1, ..., e_n\}$ and $\{u_1, \dots, u_m\}$ are the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

19. (a) Prove that any reparametrization of a regular curve is regular.

(b) Let U and $\tilde{U}$ be open subsets of $\mathbb{R}^2$ and let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch. Let $\Phi : \tilde{U} \to U$ be a bijective smooth map with smooth inverse map $\Phi^{-1} : U \to \tilde{U}$. Then, prove that $\bar{\sigma} = \sigma \circ \Phi : \tilde{U} \to \mathbb{R}^3$ is a regular surface patch.

20. Prove that a local diffeomorphism $f : S_1 \to S_2$ is conformal if and only if there is a function $\lambda : S_1 \to \mathbb{R}$ such that :

$$f^* \langle v, w \rangle_p = \lambda(p) \langle v, w \rangle_p$$

for all $p \in S_1$ and $v,\ w \in T_p S_1$.

Turn over

21. (a) Prove that a smooth map $f : S_1 - S_2$ is a local isometry if and only if the symmetric bilinear forms $\langle , \rangle_p$ and $f^*\langle , \rangle_p$ on T_pS_1 are equal for all $p \in S_1$.

(b) Let S be a (connected) surface of which every point is an umbilic. Then, prove that S is an open subset of a plane or a sphere.

(2 × 5 = 10 weightage)