D 111187 **(Pages : 2)** **Name..................................**

Reg. No...............................

THIRD SEMESTER M.Sc. (CBCSS) REGULAR/SUPPLEMENTARY DEGREE EXAMINATION, NOVEMBER 2024

Mathematics

MTH 3C 12—COMPLEX ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Which subsets of the unit sphere S correspond to the real and imaginary axes in the complex plane ?

2. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a^{n^2} z^n, a \in \mathbb{C}$.

3. Show that $\lim n^{\frac{1}{n}} = 1$.

4. Give the power series expansion of $\sqrt{z}$ about z = 1.

5. Evaluate $\int_{\gamma} \frac{e^{iz}}{z-a} dz$, where $\gamma(t) = a + re^{it}, 0 \le t \le 2\pi$.

6. Determine the type of singularity of $f(z) = \frac{\sin z}{z}$ at z = 0.

7. Find the number of zeroes of $z^7 - 4z^3 + z - 1$ enclosed by $|z| = 1$.

8. Show that a function $f : [a,b] \to \mathbb{R}$ is convex iff the set $A = \{(x,y) : a \le x \le b \text{ and } f(x) \le y\}$ is convex.

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each unit.*
Each question has weightage 2.

Unit 1

9. Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence R > 0. Then show that for each $k \ge 1$ the series $\sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)a_n (z-a)^{n-k}$ has radius of convergence R.

Turn over

10. Suppose $f : G \to \mathbb{C}$ is analytic and that G is connected. Show that if $f(z)$ is real for all z in G then f is constant.

11. Let $\gamma[a,b] \to \mathbb{R}$ be non-decreasing. Show that γ is of bounded variation and $V(\gamma) = \gamma(b) - \gamma(a)$.

UNIT 2

12. Let $f : G \to \mathbb{C}$ be analytic and suppose $\bar{B}(a;r) \subset G(r > 0)$. Show that if $\gamma(t) = a + re^{it}, 0 \le t \le 2\pi$, then $f(x) = \frac{1}{2\pi 1} \int_{\gamma} \frac{f(w)}{w - z} dw$ for $|z - a| < r$.

13. Let $\gamma(t) = 1 + e^{it}$ for $0 \le t \le 2\pi$. Find $\int_{\gamma} \left(\frac{z}{z-1}\right)^n dz$ for all positive integer n.

14. Suppose that $f : G \to \mathbb{C}$ is analytic and one-one; show that $f'(z) \ne 0$ for any z in G.

UNIT 3

15. State and prove Casorati-Weierstrass theorem.

16. State and prove Maximum Modulus Theorem.

17. Let $a < b$ and let G be the vertical strip $\{x + iy : a < x < b\}$. Suppose $f : G^{-} \to \mathbb{C}$ is continuous, f is analytic in G and $|f(z) \le 1$ for z on ∂G. Then show that $|f(z) \le 1$ for all z in G.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. (a) Let G be either the whole plane $\mathbb{C}$ or some open disk. Show that if $u : G \to \mathbb{R}$ is a harmonic function than u has a harmonic conjugate.

    (b) Find the fixed points of a dilation, a translation and the inversion on $\mathbb{C}_{\infty}$.

19. (a) Let $Sz = \frac{az + b}{cz + d}$ and $Sz = \frac{\alpha z + \beta}{\gamma z + \delta}$. Show that S = T iff there is a non-zero complex number λ such that $\alpha = \lambda a, \beta = \lambda b, \gamma = \lambda c, \delta = \lambda d$.

    (b) Find $\int_{\gamma} z^{\frac{-1}{2}} dz$ where γ is the upper half of the unit circle from + 1 to – 1.

20. State and prove Goursat's Theorem.

21. Evaluate $\int_{0}^{\infty} \frac{x^{-c}}{1 + x} dx$, when $0 < c < 1$.

(2 × 5 = 10 weightage)