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D 111192 **(Pages : 3)** **Name..................................**

Reg. No...............................

THIRD SEMESTER M.Sc. (CBCSS) REGULAR/SUPPLEMENTARY DEGREE EXAMINATION, NOVEMBER 2024

Mathematics

MTH 3E 03—MEASURE AND INTEGRATION

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Show that for very complex measureable function f on X, there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha|f|$.
2. Give an example where strict inequality occurs in Fatou's Lemma.
3. Prove $\phi(\mathrm{E}) = \int_{\mathrm{E}} s\,d\mu$ is a measure on M, where s is measurable simple function on X and $\mathrm{E} \in \mathrm{M}$.
4. Define Boreal measure, regular measure, and σ-finite measure.
5. Let μ and λ be positive measure on a σ algebra $\mathcal{M}$. Is is true that "Either μ is absolutely continuous with respect to λ or λ is absolutely continuous with respect to μ. Justify your answer.
6. Prove that the Lebesgue decomposition of a measure is unique.
7. Define measurable rectangle, elementary sets and monotone class.
8. True or False. Product of two complete measure is complete. Justify your answer.

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each unit.*
Each question has weightage 2.

Unit 1

9. State and prove Lebesgue's Dominated Convergence Theorem.
10. Prove the uniqueness of the existence of positive measure in Riesz Representation Theorem.
11. Let μ be a regular Borel measure on a compact Hausdorff space X; assume $\mu(\mathrm{X}) = 1$. Prove that there is a compact set $\mathrm{K} \subset \mathrm{X}$ (the carrier or support of μ) such that $\mu(\mathrm{K}) = 1$ but $\mu(\mathrm{H}) < 1$ for every proper compact subset H of K.

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UNIT 2

12. State and prove Vitali-Caratheodory theorem.

13. Show that if μ is complex measure on X then $|\mu|(X) < \infty$.

14. Suppose μ and λ are measure on a σ-algebra $\mathcal{M}$, μ is positive, and λ is complex. Then prove that the following two conditions are equivalent :

    (a) $\lambda \ll \mu$.

    (b) To every $\varepsilon > 0$ corresponds a $\delta > 0$ such that $|\lambda(E)| < \varepsilon$ for all $E \in \mathcal{M}$ with $\mu(E) < \delta$.

UNIT 3

15. Define the x-section and y-section of a measurable function $f(x, y)$ and show that they are measurable with respect to respective σ-algebra.

16. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \upsilon)$ be σ-finite measure space, and f be an $\mathcal{S} \times \mathcal{T}$-measurable function on $X \times Y$ write $\phi(x) = \int_Y f_x d\upsilon$ and $\chi(y) = \int_X f^y d\mu$ for $x \in X$ and $y \in Y$. Then prove ϕ is $\mathcal{S}$-measurable and χ is $\mathcal{S}$-measurable and

$$\int_X \phi d\mu = \int_Y \chi d\upsilon = \int_{X \times Y} f d(\mu \times \upsilon).$$

17. Let m_k denote Lebesgue measure on $\mathbb{R}^k$. Show that if $k = r + s, r \geq 1, s \geq 1,$ then m_k is the completion of the product measure $m_r \times m_s$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. (i) Prove that for every real valued non-negative measurable function $f(x)$ on measurable space can be approximated by non-negative monotonically increasing simple measurable functions $s_n(x)$ on measurable space which converges point wise to $f(x)$.

    (2 weightage)

    (ii) Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

    (3 weightage)

19. (i) Let X be a locally compact Hausdorff space in which every open set in σ-compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K. Then show that μ is regular.

    (3 weightage)

    (ii) Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be a linear transformation. Then prove that there exists a non-negative real number $\Delta(T)$ such that, for every $E \subset \mathcal{M}, m(T(E)) = (\Delta(T))(m(E))$.

    (2 weightage)

20. Show that if X is a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a unique regular complex Borel measure μ, in the sense that

$$\Phi f = \int_X f d\mu$$

for every $f \in C_0(X)$. Moreover, the norm of Φ is the total variation of μ :

$$\|\Phi\| = |\mu|(X).$$

21. Let $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ be two measurable space. Show that $\mathscr{S} \times \mathscr{T}$ is the smallest monotone class containing all elementary sets.

(2 × 5 = 10 weightage)

D 111192-A **(Pages : 5)** **Name..................................**

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THIRD SEMESTER M.Sc. (CBCSS) (REGULAR/SUPPLEMENTARY) DEGREE EXAMINATION, NOVEMBER 2024

Mathematics

MTH 3E 03—MEASURE AND INTEGRATION

(2019 Admission onwards)

(Multiple Choice Questions for SDE Candidates)

[Improvement Candidates need not appear for MCQ part]

Time : 20 Minutes	Total No. of Questions : 20	Maximum : 5 Weightage

INSTRUCTIONS TO THE CANDIDATE

1. This Question Paper carries Multiple Choice Questions from 1 to 20.
2. The candidate should check that the question paper supplied to him/her contains all the 20 questions in serial order.
3. Each question is provided with choices (A), (B), (C) and (D) having one correct answer. Choose the correct answer and enter it in the main answer-book.
4. The MCQ question paper will be supplied after the completion of the descriptive examination.

MTH 3E 03—MEASURE AND INTEGRATION

(Multiple Choice Questions for SDE Candidates)

1. Let $(\mathbb{R}, \tau)$ be a topological space and given that $\{1, 2, 3\}$ and $\{2, 4, 6, 8, \ldots\}$ are open sets in the given topology $(\mathbb{R}, \tau)$. Then which of the following is necessarily true ?

   (A) $\{1, 3, 4\}$ is an open set $(\mathbb{R}, \tau)$.
   (B) $\{2, 4, 6, 8, \ldots\}$ is a closed set in $(\mathbb{R}, \tau)$.
   (C) $\{2\}$ is an open set in $(\mathbb{R}, \tau)$.
   (D) $\{0, 1, 2, 4, 6, \ldots\}$ is an open set in $(\mathbb{R}, \tau)$.

2. Let X be a non-empty set, (X, τ_1) be the discrete topological space, (X, τ_2) be a topological space, then :

   (A) There exists no continuous map $f : X \to X$.
   (B) There exists continuous map $f : X \to X$ depending on τ_2.
   (C) There exists exactly one continuous map $f : X \to X$.
   (D) Any map $f : X \to X$ is continuous.

3. Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then :

   (A) $\mu(\phi) \neq 0$.
   (B) $\mu(A_1 \cup A_2 \cup \ldots \cup A_n) < \mu(A_1) + \ldots + \mu(A_n)$ if $A_1, A_2, \ldots, A_n$ are pairwise disjoint members of $\mathfrak{M}$.
   (C) $A \subset B$ implies $\mu(A) \geq \mu(B)$ if $A, B \in \mathfrak{M}$.
   (D) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \cup_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$, and $A_1 \subset A_2 \subset \ldots$

4. Let $\mathbb{Q}$ be the set of rational numbers then Lebesgue measure of $\mathbb{Q} \cap [0, 1]$ is :

   (A) 0.
   (B) 1.
   (C) 2.
   (D) ∞.

5. Let $\mathbb{I}$ be the set of irrational numbers then measure of $\mathbb{I} \cap [0, 1]$ is :

   (A) 0.
   (B) 1.
   (C) 2.
   (D) ∞.

6. If $f_n : X \to [0, \infty]$ is measurable, for each positive integer n and μ is a positive measure on σ-algebra $\mathfrak{M}$ then :

   (A) $\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu = \liminf_{n\to\infty} \int_X f_n d\mu$.
   (B) $\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \neq \liminf_{n\to\infty} \int_X f_n d\mu$.

(C) $\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu = \limsup_{n\to\infty} \int_X f_n d\mu.$

(D) $\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \le \liminf_{n\to\infty} \int_X f_n d\mu.$

7. Let $E \in \mathfrak{M}, \mu$ is a positive measure on $\mathfrak{M}$ and if $f(x) = 0$ for all $x \in E$, then :

(A) $\int_E f d\mu = 0,$ even if $\mu(E) = \infty.$

(B) $\int_E f d\mu = 0,$ only if $\mu(E) < \infty.$

(C) $\int_E f d\mu$ need not be zero.

(D) $\int_E f d\mu = 0,$ only if $\mu(E) = \infty.$

8. Let μ is a positive measure on σ-algebra $\mathcal{M}$. For $E \in \mathcal{M}$ if $f \ge 0,$ then :

(A) $\int_E f d\mu < \int_X X_E f d\mu.$

(B) $\int_E f d\mu > \int_X X_E f d\mu.$

(C) $\int_E f d\mu = \int_X X_E f d\mu.$

(D) $\int_E f d\mu \ne \int_X X_E f d\mu.$

9. Choose the false statements :

(A) A set E in a measure space (with measure μ) is said to have σ-finite measure if E is a countable union of set E, with $\mu(E_i) < \infty.$

(B) A set E in a topological space is called σ-compact if E is a union of compact sets.

(C) Every σ-compact set has σ-finite measure.

(D) If $E \in \mathfrak{M}$ and E has σ-finite measure, then E is inner regular.

10. If $A \subset \mathbb{R}^1$ and every subset of A is Lebesgue measurable then :

(A) $m(A) = \infty$.

(B) $m(A) = 0$.

(C) $0 < m(A) < \infty$.

(D) $m(A) > 1$.

11. I. Every set of positive measure has a non-measurable subset.

II. Let E be the cantor set, $E \subset \mathbb{R}^1$ then $m(E) = 0$, where m is the Lebesgue measure.

(A) Both I and II are true.

(B) Both I and II are false.

(C) I is true and II is false.

(D) I is false and II is true.

12. Let $\mathfrak{M}$ be a σ-algebra in a set X and $E \in \mathfrak{M},$ then :

(A) $|\mu|(E) \ne |\mu(E)|.$

(B) $|\mu|(E) = |\mu(E)|.$

(C) $|\mu|(E) \ge |\mu(E)|.$

(D) $|\mu|(E) \le |\mu(E)|.$

13. If μ is a positive measure and $|\mu|$ is the total variation of μ, then :

(A) $|\mu| \neq \mu$.
(B) $|\mu| = \mu$.
(C) $|\mu| < \mu$.
(D) $|\mu| > \mu$.

14. Let X be a set, $\mathfrak{M}$ be a σ-algebra on X and μ be a real measure on $\mathfrak{M}$. For every partition (E_i) of any set $E \in \mathfrak{M}$, define $|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|$. Then choose the incorrect statement :

(A) $\frac{|\mu|+\mu}{2}$ is a positive measure on $\mathfrak{M}$.

(B) $\frac{|\mu|-\mu}{2}$ need not be a positive measure on $\mathfrak{M}$.

(C) $\frac{|\mu|+\mu}{2}$ is called positive variation of μ.

(D) $\frac{|\mu|-\mu}{2}$ is called negative variation of μ.

15. Let λ_1, λ_2 are two measures on a σ-algebra $\mathfrak{M}$ and A, B be two disjoint sets in $\mathfrak{M}$. Also given that λ_1 is concentrated on A and λ_2 is concentrated on B. Then :

I. λ_1 and λ_2 are mutually singular.

II. $|\lambda_1|$ and $|\lambda_2|$ are mutually singular.

(A) Both I and II are true.
(B) Both I and II are false.
(C) I is true and II is false.
(D) I is false and II is true.

16. Suppose μ is positive measure on a σ-algebra $\mathfrak{M}$, $g \in L^1(\mu)$, and $\lambda(E) = \int_E g\, d\mu$ for $E \in \mathfrak{M}$. Then :

(A) $|\lambda|(E) \neq \int_E |g|\, d\mu$, for $E \in \mathfrak{M}$.

(B) $|\lambda|(E) \leq \int_E |g|\, d\mu$, for $E \in \mathfrak{M}$.

(C) $|\lambda|(E) \geq \int_E |g|\, d\mu$, for $E \in \mathfrak{M}$.

(D) $|\lambda|(E) = \int_E |g|\, d\mu$, for $E \in \mathfrak{M}$.

17. Let μ be a positive measure, suppose $1 \le p \le \infty$, and let q be the exponent conjugate to p. If $g \in L^q(\mu)$ and $\phi_g(f) = \int_X fg \, d\mu$ then choose incorrect statement :

    (A) ϕ_g is bounded linear functional on $L^p(\mu)$.

    (B) ϕ_g has norm atmost $\|g\|_q$.

    (C) For $p = \infty$ all bounded linear functionals on $L^p(\mu)$ have the form $\phi_g(f) = \int_X fg \, d\mu$.

    (D) For $1 < p < \infty$ all bounded linear functionals on $L^p(\mu)$ have the form $\phi_g(f) = \int_X fg \, d\mu$.

18. Let X be a locally compact Hausdorff space. Choose the incorrect statement :

    (A) $C_c(X)$ is a dense subspace $C_0(X)$, relative to supnorm.

    (B) Bounded linear functional on $C_c(X)$, has an extention to a bounded linear fucntion on $C_0(X)$.

    (C) Bounded linear functional on $C_c(X)$, has more than two extension to a bounded linear function on $C_0(X)$.

    (D) $C_0(X)$ is a Banach space.

19. Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{J}, \lambda)$ are σ-finite measure spaces. Suppose that $Q \in \mathscr{S} \times \mathscr{J}$ and define $\phi(x) = \lambda(Q_x), \psi(y) = \mu(Q^y)$ for all $x \in X, y \in Y$. Then :

    (A) ϕ is $\mathscr{J}$-measurable and ψ is $\mathscr{J}$-measurable.

    (B) ϕ is $\mathscr{J}$-measurable and ψ is $\mathscr{S}$-measurable.

    (C) ϕ is $\mathscr{S}$-measurable and ψ is $\mathscr{J}$-measurable.

    (D) ϕ is $\mathscr{S}$-measurable and ψ is $\mathscr{S}$-measurable.

20. Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{J}, \lambda)$ are σ-finite measure spaces. Suppose that $Q \in \mathscr{S} \times \mathscr{J}$ and define $(\mu \times \lambda)(Q) = \int_X \lambda(Q_x) d\mu(x) = \int_X \mu(Q^y) d\lambda(y)$.

    I : $\mu \times \lambda$ is σ-finite.

    II : $\mu \times \lambda$ countable subadditive not countable additive on $\mathscr{S} \times \mathscr{J}$.

    (A) Both I and II are true. (B) Both I and II are false.

    (C) I is true and II is false. (D) I is false and II is true.