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FOURTH SEMESTER M.Sc. (CBCSS) [REGULAR/SUPPLEMENTARY] DEGREE EXAMINATION, APRIL 2025

Mathematics

MTH4C15—ADVANCED FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum Weightage : 30

Part A

Answer **all** *the questions.*
Each question carries 1 weightage.

1. Let A be a bounded operator on a Banach space X. Prove that every $\lambda \in C$ with $|\lambda| \geq \|A\|$ is a regular point of the operator A.
2. Let A be a self-adjoint bounded operator on a Hilbert space H. Prove that two eigen vectors corresponding to distinct eigen values λ_1, λ_2 of the operator A are orthogonal.
3. Let E be a linear space. Let $T : E \to E$ be any operator $E_1 + E_2 = E$ and let P be the projection onto E_1 parallel to E_2. Show that TP = PT iff E_1 and E_2 are invariant subspaces of T.
4. Show that every orthogonal projection P on a Hilbert space H satisfies $O \leq P \leq I$.
5. Define spectral family for a self-adjoint bounded operator on a Hilbert space H.
6. Prove that a closed convex set in a Banach space is perfectly convex.
7. Let X be a normed space, let $E_0 \hookrightarrow X$ be a subspace of X and let $f_0 \in E_0^*$. Show that there exists $f \in X^*$ such that $f / E_0 = f_0$ and $\|f\|_{X^*} = \|f_0\|_{E^*}$.
8. Prove that if the element *e-yx* of a Banach algebra is invertible then the element *e-xy* is invertible.

(8 × 1 = 8 weightage)

Part B

Answer **six** *questions by choosing any* **two** *questions from each unit.*
Each question carries a weightage 2.

Unit I

9. Let $A : L_2[0,1] \to L_2[0,1]$ be defined by $(Ax)t = t \,.\, x(t), t \varepsilon [0,1]$. Determine the spectrum of A.

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10. Let X be an infinite dimensional Banach space. Show that the identity operator $I : X \to X$ is not compact.

11. Let A be in L(H), where H is a Hilbert space. Define the operator B on $H^2 = H \oplus H$ by $B = \begin{pmatrix} 0 & iA \\ -iA^* & 0 \end{pmatrix}$. Prove that B is self adjoint and find $\|B\|$.

Unit II

12. Let A be a bounded operator on a Hilbert space H. Show that if A is symmetric and $A \geq 0$, then for any polynomial $p(\lambda)$ with non-negative co-efficients we have $p(A) \geq 0$.

13. Let P_1, P_2 be two orthogonal projections on a Hilbert space H such that $P_1P_2 = P_2P_1 = P$. Show that P is an orthogonal projection and that $\operatorname{Im} P = E_1 \cap E_2$ where $E_i = P_i H$ for $i = 1, 2$.

14. State Hilbert theorem on the spectral decomposition of self adjoint bounded operators.

Unit III

15. Let E_1, E_2 be closed subspaces of a Banach space X with $E_1 \cap E_2 = 0$ and $E_1 + E_2 = X$. Show that the projection $P : X \to E_1$, parallel to E_2 is a bounded operator.

16. Let X be a Banach space. Show that if X* is separable, then X is also separable.

17. Let $\mathcal{A}$ be a Banach algebra. Prove that if $X \, \varepsilon \, \mathcal{A}$, $\|x\| < 1$ then $\left\|(e-x)^{-1} - e - x\right\| \leq \dfrac{\|x\|^2}{1-\|x\|}$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage 5.

18. (a) Show that for every $\in > 0$, there is only a finite number of linearly independent eigen vectors corresponding to eigen values λ_i with $|\lambda_i| \geq \in$.

(b) Prove that every complete metric space is a set of second category.

19. (a) Let H be a Hilbert space and let $A : H \to H$ be a bounded operator on H. Prove that $\langle Ax, x \rangle \, \varepsilon \, R$ if and only if A is symmetric.

(b) State and prove closed graph theorem.

20. (a) Let A, B be two bounded operators on a Hilbert space H such that $A \geq 0$, $B \geq 0$ and $AB = BA$. Show that $AB \geq 0$.

(b) State and prove Banach-Steinhans theorem.

21. Let K be a closed convex set in a Banach space X and let $x_0 \notin k$. Show that there exists $f \, \varepsilon \, X^*$ such that $f(x_0) > \sup_{x \in k} f(x)$.

(2 × 5 = 10 weightage)