D 121286 **(Pages : 2)** **Name..................................**

Reg. No...............................

FOURTH SEMESTER M.Sc. (CBCSS) (REGULAR/SUPPLEMENTARY) DEGREE EXAMINATION, APRIL 2025

Mathematics

MTH4E08—COMMUTATIVE ALGEBRA

(2019 Admission onwards)

Time : Three Hours Maximum Weightage : 30

Part A

Answer **all** *questions.*
Each questions carries a weightage of 1.

1. Verify whether $f(x) = x + 1$ is a homomorphism of the ring $\mathbb{Z}$ of integers.
2. Find all units in the ring $\mathbb{Z}_{10}$ of integers mod 10.
3. Find the nil radical of the ring $\mathbb{Z}$ of integers.
4. Verify whether the set {1, 3, 5, 7} is a multiplicatively closed subset of the ring $\mathbb{Z}_{10}$.
5. Let S be a multiplicatively closed subset of a ring A and let $0 \in S$. Show that $S^{-1}A$ is the zero ring.
6. Let M be an A-module and S be a multiplicatively closed subset of A. Describe the module $S^{-1}M$.
7. Verify whether $\frac{1}{2}$ is integral over $\mathbb{Z}$ in the field of rationals.
8. Verify whether the ring $\mathbb{Z}$ of integers satisfy d.c.c. on ideals.

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each module.*
Each question carries a weightage of 2.

Module I

9. Let a be an ideal of commutative ring A. Show that there is a one to one correspondence beteen the ideals of A/a and the ideals of A which contain a.
10. Let A be a field and B be a non-zero ring. Show that every homomorphism $f : A \to B$ is injective.
11. Show that the set of all nilpotent elements of a ring A is an ideal of A.

Module II

12. Let S be a multiplicatively closed subset of a ring A. Define ~ on A × S by

$$(a,s) \sim (b,t) \text{ if } (at - bs)u = 0 \text{ for some } u \in S.$$

Show that ~ is an equivalence relation on A × S.

13. Show that if $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is an exact sequence of A-modules then $S^{-1}M' \xrightarrow{S^{-1}\alpha} S^{-1}M \xrightarrow{S^{-1}\beta} S^{-1}M''$ is also exact.

14. Let M be an A-module such that $M_m = 0$ for all maximal ideals m of A. Show that M = 0.

Module III

15. Let A be a subring of a ring B and let $x \in$ B be integral over A. Show that A[x] is a finitely generated A-module.

16. Let A be a subring of B and b be an ideal of B. Let $a = b \cap A$. Show that if B is integral over A then B/b is integral over A/a.

17. Let $0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$ be an exact sequence of A-modules. Show that if M' and M'' are Noetheran then M is Neotherian.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (a) Define Jacobson radical of a ring.
    (b) Let $\mathcal{R}$ be the Jacobson radical of a ring. Show that $x \in \mathcal{R}$ if and only if $1 - xy$ is a unit in A for every $y \in$ A.

19. Let S be a multiplicatively closed set in a ring A and let S^{-1} A be the ring of fractions. Show that
    (a) Every ideal of S^{-1}A is an extended ideal.
    (b) The prime ideals of S^{-1}A are in one to one correspondence with the prime ideals of A which do not meet S.

20. Let q be a p-primary ideal of a ring A and let $x \in$ A. Show that
    (a) if $x \in q$ then $(q : x) = (1)$.
    (b) if $x \notin q$ then $(q : x)$ is p-primary.
    (c) if $x \notin q$ then $(q : x) = q$.

21. (a) Define Noetherian ring.
    (b) Show that if A is Noetherian then the polynomial ring A[x] is Noetherian.
    (c) Prove that in a Noetherian ring every irreducible ideal is primary.

(2 × 5 = 10 weightage)