D 121287 **(Pages : 3)** **Name..................................**

Reg. No...............................

FOURTH SEMESTER M.Sc. (CBCSS) REGULAR/SUPPLEMENTARY DEGREE EXAMINATION, APRIL 2025

Mathematics

MTH4E09—DIFFERENTIAL GEOMETRY

(2019 Admission onwards)

Time : Three Hours Maximum Weightage : 30

Part A

Answer **all** *questions.*
Each questions carries a weightage of 1.

1. Sketch the level set $f^{-1}(c)$ at $c = 1$ for the function $f(x_1, x_2) = x_2$.
2. Define the gradient ∇f of a smooth function $f : \mathrm{U} \to \mathbb{R}$ where U is open in $\mathbb{R}^{n+1}$.
3. Show that the 2-sphere $x_1^2 + x_2^2 + x_3^2 = 1$ is connected.
4. For a parametrized curve α given by $\alpha(t) = (t, t^2)$ find the velocity vector at $t = 1$.
5. Let S be an n-surface which contains a straight line segment $\alpha(t) = p + tv$. Show that α is a geodesic in S.
6. Sketch a Euclidean parallel vector field on a parametrized curve α.
7. Find the normal curvature $k(v)$ of the sphere S given by $x_1^2 + x_2^2 + x_3^3 = r^2$ at a point $p \in$ S.
8. Define parametrized n-surface.

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each unit.*
Each question carries a weightage of 2.

Unit I

9. Show that the set of all vectors at a point p in $\mathbb{R}^n$ form a vector space.
10. Let $f : \mathrm{U} \to \mathbb{R}$ be a smooth function where $\mathrm{U} \subseteq \mathbb{R}^{n+1}$. Show that the gradient at $p \in f^{-1}(c)$ is orthogonal to all vectors tagent to $f^{-1}(c)$ at p.
11. Show that the n-sphere $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1$ is an n-surface in $\mathbb{R}^{n+1}$.

Unit II

12. Let $\bar{X}, \bar{Y}$ be smooth vector fields along a parametrized curve. Show that $\left(\bar{X}+\bar{Y}\right)=\bar{X}+\bar{Y}$.

13. Show that the covarient derivatve $\bar{X}'$ on a n-surface S is independent of the orientation on S.

14. Let C be a connected oriented plane curve and $\beta : I \to C$ be a unit speed parametrization of C. Show that β is either one to oe or periodic.

Unit III

15. Let $k(v)$ be the normal curvature and $k_1(p), k_2(p), \ldots, k_n(p)$ be the principal curvatures with directions $v_1, v_2, \ldots, v_n$ of a n-surface. Show that $k(v) = \sum_{i=1}^{n} k_i(p)\left(v \cdot v_i\right)^2$.

16. Show that for $\psi : \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$\psi(\theta, \phi) = (\cos\theta, \sin\theta, \cos\phi, \sin\phi)$$

the image of ψ is the cartesian product of two circles.

17. State the conditions on an n-surface S so that the Gauss map $N : S \to S^n$ is a diffeomorphism.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (a) Define integral curve of a vector field.

(b) Let $\bar{X}$ be a smooth vector field on an open set $U \subseteq \mathbb{R}^{n+1}$ and $p \in U$. Show that there exists an open interval I containing 0 and an integral curve $\alpha : I \to U$ of $\bar{X}$ such that $\alpha(0) = p$.

19. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ given as $f^{-1}(c)$ where $\nabla_f(p) \neq 0$ for all $p \in S$. Let $v = \left(a_1, a_2, \ldots, a_{n+1}\right) \in S$. Prove that

(a) if $g : \mathbb{R}^{n+1} \to \mathbb{R}$ is defined by

$$g\left(x_1, x_2, \ldots, x_{n+1}\right) = a_1 x_1 + a_2 x_2 + \ldots + a_{n+1} x_{n+1}$$

then $g^{-1}(c)$ is an n-plane parallel to $v^{\perp}$.

(b) There exists a point $p \in S$ such that $N(p) = \pm v$.

20. (a) Define differential 1-form.

(b) Let $\bar{X}$ be a smooth vector field on an open set $U \subseteq \mathbb{R}^{n+1}$. Show that

(i) if $w_{\bar{X}}$ is defined by

$$w_{\bar{X}}(p,v) = \bar{X}(p) \cdot (p,v) \text{ for } p \in U, v \in \mathbb{R}^{n+1}$$

then $w_{\bar{X}}$ is a 1-form on U.

(ii) for each 1-form w on U there exists a unique choice of functions $f_i : U \to \mathbb{R}$ such that

$$w = \sum_{i=1}^{n+1} f_i dx_i.$$

21. (a) Define the second fundamental form of an oriented n-surface.

(b) Show that on each compact oriented n-surface S there exists a point $p \in S$ such that the second fundamental form at p is definite.

(2 × 5 = 10 weightage)