D 131314 **(Pages : 3)** Name.......................................

Reg. No...................................

FIRST SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2025

(CBCSS)

Mathematics

MTH 1C 02—LINEAR ALGEBRA

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question carries a weightage 1.

1. Is $W = \left\{(a_1, \ldots, a_n)/a_2 = a_1{}^2\right\}$ a subspace of $\mathbb{R}^n$? Justify your answer.

2. Let F be a field and let T be the operator on F^2 defined by $T(x, y) = (x, 0)$. Find $[T]_B$, where B be the standard ordered basis for F^2.

3. Let V be a vector space and let V^* be the collection of all linear functionals on V . Show that $\dim V^* = \dim V$.

4. Find the null space, nullity, range space and rank for the zero transformation and the identity transformation on a finite- dimensional space V .

5. What is meant by minimal polynomial for a linear operator T on a finite dimensional space V over the field F.

6. Let V be a vector space and E be a projection of V. Prove that a vector β is in range of E if and only if $E\beta = \beta$.

7. Let W be a subspace of an inner product space V and let β be a vector in V. If a best approximation to vectors in W exists, then show that it is unique.

8. Show that the vector (x, y) in $\mathbb{R}^2$ is orthogonal to $(-y, x)$ with respect to standard inner product.

(8 × 1 = 8 weightage)

Turn over

Part B (Paragraph Type Questions)

Answer any **two** *questions, choosing* **two** *questions from each module.*
Each question carries a weightage 2.

MODULE I

9. Let W_1 and W_2 be subspaces of a vector space V such that the set -theoretic union of W_1 and W_2 is also a subspace. Prove that one of the subspaces W_1 or W_2 is contained in the other.

10. If W is a proper subspace of a finite dimensional vector space V, then show that W is finite dimensional and $\dim W < \dim V$.

11. Show that every n -dimensional vector space over the field F is isomorphic to the space F^n.

MODULE II

12. If W_1 and W_2 are subspaces of a finite dimensional vector space, then show that $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.

13. Let $T : R^2 \to R^2$ be the linear transformation defined by $T(x, y) = (3x + 4y, 2x - 5y)$.

    Find $[T]_S$ when (i) $S = \{(1, 0), (0, 1)\}$; and (ii) $S = \{(1, 2), (2, 3)\}$.

14. If W is an invariant subspace for T, then show that W is invariant under every polynomial in T. Thus show that the conductor $S(\alpha ; W)$ is an ideal in the polynomial algebra $F[x]$, for each α in V.

MODULE III

15. Let T be a linear operator on a finite-dimensional space V. If T is diagonalizable and if $c_1, \ldots, c_k$ are the distinct characteristic values of T, then show that there exist linear operators $E_1, \ldots, E_k$ on V such that

    (i) $T = c_1E_1 + \ldots + c_kE_k$;

    (ii) $I = E_1 + \ldots + E_k$;

    (iii) $E_iE_j = 0, i \neq j$;

(iv) $E_i^2 = E_i$ (E_i is a projection);

(v) the range of E_i is the characteristic space for T associated with c_i.

16. Prove that an orthogonal set of non-zero vectors is linearly independent.

17. Let V be an inner product space, W a finite-dimensional subspace, and E the orthogonal projection of V on W. Then show that the mapping $\beta \to \beta - E\beta$ is the orthogonal projection of V on $W^{\perp}$.

(6 × 2 = 12 weightage)

Part C(Essay Type Questions)

Answer any **two** *questions.*
Each question carries a weightage 5.

18. If W_1 and W_1 are finite dimensional subspace of a vector space V, then prove that $W_1 + W_2$ is finite dimensional and $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$.

19. Let V and W be finite-dimensional vector spaces over the field F. Let B be an ordered basis for V with dual basis B^*, and let B' be an ordered basis for W with dual basis B'^*. Let T be a linear transformation from V into W ; let A be the matrix of T relative to B, B' and let C be the matrix of T^1 relative to B'^*, B^*. Then show that $C_{ij} = A_{ji}$.

20. Let $g, f_1, ..., f_r$ be linear functionals on a vector space V with respective null spaces $N, N_1, ..., N_r$. Then show that g is a linear combination of $f_1, ..., f_r$ if and only if N contains the intersection $N_1 \cap .... \cap N_r$.

21. Let W be a subspace of an inner product space V and let $\beta \in V$. Then prove that,

(i) The vector $\alpha \in W$ is a best approximation to $\beta \in V$ by vectors in W if and only if $\boldsymbol{\beta} - \boldsymbol{\alpha}$ is orthogonal to every vector in W.

(ii) If a best approximation to $\beta \in V$ by vectors in W exists, it is unique.

(ii) If W is finite-dimensional and $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is any orthonormal basis for W, then the vector $\alpha = \sum_{k=1}^{n} \frac{(\beta/\alpha_k)}{\|\alpha_k\|^2} \alpha_k$ is the (unique) best approximation to $\boldsymbol{\beta}$ by vectors in W.

(2 × 5 = 10 weightage)