D 131315 **(Pages : 4)** **Name.....................................**

Reg. No.................................

FIRST SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2025

(CBCSS)

Mathematics

MTH1C03—REAL ANALYSIS—I

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A (Short Answer Type Questions)

Answer **all** *questions.*

Each question carries a weightage 1.

1. Examine whether the following subset of $\mathbb{R}^2$ is : (i) Closed ; (ii) Open (iii) Perfect ; and (iii) bounded.

   The set of all complex numbers z such that $|z| \leq 1$.

2. If X is a metric space and $E \subset X$, then prove that $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

3. Discuss the continuity/discontinuity behaviour of the function f defined by :

$$f(x) = \begin{cases} x+2 & (-3 < x < -2) \\ -x-2 & (-2 \leq x < 0) \\ x+2 & (0 \leq x < 1) \end{cases}$$

4. Define separated subsets of a metric space.

5. Suppose f is differentiable in (a, b) and $f'(x) \leq 0$ for all $x \in (a, b)$, then prove that f is monotonically decreasing.

6. If $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on $[a, b]$, then prove that $fg \in \Re(\alpha)$

Turn over

7. Define equicontinuous family of functions.

8. Define rectifiable curve.

(8 × 1 = 8 weightage)

Part B (Paragraph Type Questions)

Answer any **two** *questions from each module.*

Each question carries a weightage 2.

Module I

9. If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}^1$, such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \ldots)$, then prove that $\bigcap_{n=1}^{\infty} I_n$ is not empty.

10 Prove that every k-cell is compact.

11. Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then prove that $f(X)$ is compact.

Module II

12. Let f be defined by :

$$f(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

Find $f'(x)$ (for $x \neq 0$). Also prove that $f'(0)$ does not exist.

13. If P* is a refinement of P, then prove that :

$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

14. If $f_1 \in \Re(\alpha)$ and $f_2 \in \Re(\alpha)$ on $[a, b]$, then prove that $f_1 + f_2 \in \Re(\alpha)$, and

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha.$$

Module II

15. If γ' is continuous on $[a, b]$, then prove that γ is rectifiable, and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

16. Give an example of an everywhere discontinuous limit function, which is not Riemann-integrable.

17. If K is compact, if $f_n \in C(K)$ for $n = 1, 2, 3, \ldots$, and $\{f_n\}$ is pointwise bounded and equicontinuous on K, then prove that $\{f_n\}$ is uniformly bounded on K.

(6 × 2 = 12 weightage)

Part C (Essay Type Questions)

Answer any **two** *questions.*

Each question carries a weightage 5.

18. Let $\{E_n\}, n = 1, 2, 3, \ldots,$ be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then prove that S is countable.

19. Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n-1)}(t)$ exists for any $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then prove that there exists a point x between α and β such that :

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n.$$

20. Prove that $f \in \Re(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that :

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

21. Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \Re(\alpha)$ on $[a, b]$, for n = 1, 2, 3 and suppose $f_n \to f$ uniformly on $[a, b]$. Then prove that $f \in \Re(\alpha)$ on $[a, b]$, and

$$\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha.$$

(2 × 5 = 10 weightage)