D 132040 **(Pages : 3)** **Name..................................**

Reg. No..............................

THIRD SEMESTER M.Sc. DEGREE [REGULAR/SUPPLEMENTARY] EXAMINATION, NOVEMBER 2025

(CBCSS)

Mathematics

MTH3C13—FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*

Each question has weightage 1.

1. Define norm on a linear space. Give an example of a norm on l_1.

2. If O is an open set, then prove that the set $F = O^c$ is closed.

3. State and prove parallelogram law in the case of an inner product space.

4. Prove that for any orthonormal system $\{e_i\}_{i \geq 1} \subset H$, and for every $x \in H$,

$$\sum_{i \geq 1} \left|\langle x, e_i \rangle\right|^2 \leq \|x\|^2.$$

5. Let H be an inner product space. Describe all pairs of vectors x, y for which

$$\|x + y\| = \|x\| + \|y\|.$$

6. For non-zero linear functionals f, g and kerf = kerg, show that there exists $\lambda \neq 0$ such that $\lambda f = g$.

Turn over

7. Let L be a closed subspace. Consider the subspaces $\mathrm{L}^{\perp} \hookleftarrow \mathrm{X}^*$ and

$$\left(\mathrm{L}^{\perp}\right)_{\perp} = \left\{x \in \mathrm{X} : f(x) = 0, \text{ for all } f \in \mathrm{L}^{\perp}\right\}.$$

Then, proveb that

$$\left(\mathrm{L}^{\perp}\right)_{\perp} = \mathrm{L}.$$

8. State and prove the Banach open map theorem.

(8 × 1 = 8 weightage)

Part B

Answer **six** *questions choosing two from each unit.*

Each question has weightage 2.

Unit I

9. Let $1 < p < \infty$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all functions f, g on an interval $[a, b]$ such that the integrals $\int_a^b |f(t)|^p \, dt$, $\int_a^b |g(t)|^q \, dt$ and $\int_a^b |f(t)\, g(t)| \, dt$ exist prove that

$$\int_a^b |f(t)\, g(t)| \, dt \le \left(\int_a^b |f(t)|^p\right)^{1/p} \left(\int_a^b |g(t)|^q\right)^{1/q}$$

10. Prove that C $[a, b]$ is complete.

11. Is a quotient space a normed space ? Justify your answer.

Unit II

12. Prove that the system $\left\{\frac{1}{\sqrt{2\pi}} e^{int}\right\}_{-\infty}^{\infty}$ is an orthonormal basis of $\mathrm{L}_2[-\pi, \pi]$.

13. If E is a closed subspace of H and codim E = 1, then prove that the subspace $E^{\perp}$ is 1-dimensional.
14. Prove that f is a bounded functional if and only if f is a continuous functional.

Unit III

15. Prove that for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1, (l_p)^* = l_{p^*}$.

16. Prove the set K (X $\mapsto$ Y) of compact operators from X to Y is a linear sub-space of L (X $\mapsto$ Y).

17. If $\|A\| < 1$, then prove that (I – A) is invertible and $(I - A)^{-1} = \sum_0^\infty A^k$.

(6 × 2 = 12 weightage)

Part C

Answer **two** *questions.*

Each question has weightage 5.

18. Let E, X_i for i = 1, 2 be linear normed linear spaces with X_i being complete and $T_i : E \mapsto X_i$ isometries into dense subspaces of X_i for i = 1, 2. Then prove that the natural mapping $T_2 \circ T_1^{-1} : T_1E \mapsto T_2E$ can be extended to an isometry between X_1 and X2.
19. State and prove necessary condition for a Hilbert space to have an orthonormal basis.
20. Let E be an n-dimensional normed space. Prove that E is complete.
21. Prove that M is relatively compact if and only if for every $\varepsilon > 0$ there exists a finite ε-net in M.

(2 × 5 = 10 weightage)