**D 132044** **(Pages : 3)** **Name...................................**

**Reg. No...............................**

**THIRD SEMESTER M.Sc. DEGREE [REGULAR/SUPPLEMENTARY] EXAMINATION, NOVEMBER 2025**

(CBCSS)

Mathematics

MTH3E03—MEASURE AND INTEGRATION

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

**Part A**

*Answer* **all** *questions.*

*Each question has weightage* 1.

1. Show that $m^*E = 0$ then E is measurable.

2. Let $f_n : X \mapsto [-\infty, \infty]$ is measurable , for $n = 1, 2, 3$ .... Show that $\limsup_{n \to \infty} f_n$ is measurable.

3. Show that if $f \in L^1(\mu)$ then $f$ is finite valued almost every where.

4. Define Borel measure, regular measure, and $\sigma$-finite measure.

5. Define total variation measure.

6. If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$ then prove that $\lambda_1 \perp \lambda_2$. Also prove that if $\lambda \ll \mu$ then $|\lambda| \ll \mu$.

7. Prove that for every every $\sigma$ finite measure on a $\sigma$ algebra $\mathcal{M}$ in a set X there exists a integrable function on X which takes the value on (0,1) for every $x \in X$.

8. Define measurable rectangle, elementary sets and monotone class.

(8 × 1 = 8 weightage)

**Turn over**

## Part B

*Answer any* **two** *questions from each unit.*
*Each question has weightage* 2.

### Unit 1

9. State and prove Lebesuge's monotone convergence theorem.

10. Suppose $f$ and $g$ in $L^1(\mu)$ and $\alpha$ and $\beta$ are complex numbers. Show that $\alpha f + \beta g \in L^1(\mu)$ and $\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$.

11. State and prove Urysohn's Lemma.

### Unit 2

12. Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be a linear transformation. Then prove that there exists a nonnegative real number $\Delta(T)$ such that, for every $E \subset \mathcal{M}$,

    (a) $T(E) \in \mathcal{M}$,

    (b) $m(T(E)) = (\Delta(T))(m(E))$.

13. Show that every measurable function defined on locally compact Hausdorff space X can be approximated almost everywhere by compactly supported continuous function on X.

14. State and prove Hanh Decomposition theorem.

### Unit 3

15. Define the $x$-section and $y$-section of subset of Cartesian product of two $\sigma$- algebras and show that they are measurable with respect to respective $\sigma$-algebra.

16. Let $(X, \mathscr{G}, \mu)$ and $(Y, \mathscr{F}, \nu)$ be $\sigma$- finite measure space, and $f$ be an $\mathscr{G} \times \mathscr{F}$- measurable function on X × Y write $\phi(x) = \int_Y f_x d\nu$ and $\chi(y) = \int_X f^y d\mu$ for $x \in X$ and $y \in Y$. Then prove $\phi$ is $\mathscr{G}$-measurable and $\chi$ is $\mathscr{F}$-measurable and

$$\int_X \phi d\mu = \int_Y \chi d\nu = \int_{X \times Y} f d(\mu + \nu).$$

17. Let $m_k$ denote Lebesgue measure on $\mathbb{R}^k$. Show that if $k = r + s$, $r \geq 1$, $s \geq 1$, then $m_k$ is the completion of the product measure $m_r \times m_s$.

(6 × 2 = 12 weightage)

## Part C

*Answer any* **two** *questions.*
*Each question has weightage* 5.

18. Let $(X, \mathcal{M}, \mu)$ be a measure space, let $\mathcal{M}^*$ be the collection of all $E \subset X$ for which there exists set A and B such that $A \subset E \subset B$ and $\mu(B/A) = 0$. Prove that $\mathcal{M}^*$ is a $\sigma$ algebra containing $\mathcal{M}$. Also prove that there is unique measure $\mu^*$ on $\mathcal{M}^*$. such that $\mu^*|_{\mathcal{M}} = \mu$.

19. State and prove the theorem of Lebesuge-Radon-Nikodym.

20. Let $1 \leq p < \infty$, $\mu$ is a $\sigma$ finite measure on a measurable space X and let $\phi$ be a bounded linear functional on $L^p(\mu)$. Prove that there is a unique $g \in L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\phi(f) = \int_X fg \, d\mu$$

21. (i) State and prove Fubini theorem. (3 weightage)

(ii) Let

$$f(x, y) = \begin{cases} \dfrac{xy}{\left(x^2 + y^2\right)^2}, & -1 \leq x, y \leq 1, (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Show that the iterated integrals of $f$ over the square are equal but that $f$ is not integrable.

(2 weightage)

(2 × 5 = 10 weightage)

**D 132044–A** **(Pages : 7)** **Name.......................................**

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**THIRD SEMESTER M.Sc. DEGREE [REGULAR/SUPPLEMENTARY] EXAMINATION, NOVEMBER 2025**

(CBCSS)

Mathematics

MTH3E03—MEASURE AND INTEGRATION

(2019 Admission onwards)

[Improvement Candidates need not appear for MCQ Part]

(Multiple Choice Questions for SDE Candidates)

| **Time : 20 Minutes** | **Total No. of Questions : 20** | **Maximum : 5 Weightage** |
|---|---|---|

## INSTRUCTIONS TO THE CANDIDATE

1. This Question Paper carries Multiple Choice Questions from 1 to 20.
2. The candidate should check that the question paper supplied to him/her contains all the 20 questions in serial order.
3. Each question is provided with choices (A), (B), (C) and (D) having one correct answer. Choose the correct answer and enter it in the main answer-book.
4. The MCQ question paper will be supplied after the completion of the descriptive examination.

MTH3E03—MEASURE AND INTEGRATION

(Multiple Choice Questions for SDE Candidates)

1. Let X and Y be two topological spaces and $f : X \to Y$ be a continuous map then which of the following is necessarily true ?

   (A) $f^{-1}(V)$ is an open set in X for every closed set V in Y.

   (B) $f(V)$ is an open set in Y for every open set V in X.

   (C) $f^{-1}(V)$ is an open set in X for every open set V in Y.

   (D) None of the above options are true.

2. Let X be a non-empty set, $(X, \mathcal{T}_1)$ be the discrete topological space, $(X, \mathcal{T}_2)$ be a topological space, then :

   (A) There exists no continuous map $f : X \to X$.

   (B) There exists continuous map $f : X \to X$ depending on $\mathcal{T}_2$.

   (C) There exists exactly one continuous map $f : X \to X$.

   (D) Any map $f : X \to X$ is continuous.

3. Let $\mu$ be a positive measure on a $\sigma$–algebra $\mathfrak{M}$. Then

   (A) $\mu(\phi) \neq 0$.

   (B) $\mu(A_1 \cup A_2 \cup \ldots \cup A_n) < \mu(A_1) + \ldots + \mu(A_n)$ if $A_1, A_2, \ldots, A_n$ are pairwise disjoint members of $\mathfrak{M}$.

   (C) $A \subset B$ implies $\mu(A) \geq \mu(B)$ if $A, B \in \mathfrak{M}$.

   (D) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n$, $A_n, \in \mathfrak{M}$, and $A_1 \subset A_2 \subset \ldots$.

4. Let $\mathbb{Q}$ be the set of all rational numbers then Lebesgue measure of $\mathbb{Q} \cap [0, 1]$ is :

(A) 0. (B) 1.

(C) 2. (D) $\infty$.

5. Let $A, B \in \mathfrak{M}$ and $A \subset B$, also given that $\mu(B) = 1$ Then which of the following are necessarily true ?

(A) $\mu(A) = 1$. (B) $\mu(A) > 1$.

(C) $0 \leq \mu(A) < 1$. (D) $0 \leq \mu(A) \leq 1$.

6. If $f_n : X \to [0, \infty]$ is measurable, for each positive integer $n$ and $\mu$ is a positive measure on $\sigma$-algebra $\mathfrak{M}$ then :

(A) $\int_X \left( \lim_{n\to\infty} \inf f_n \right) d\mu = \lim_{n\to\infty} \inf \int_X f_n \, d\mu$.

(B) $\int_X \left( \lim_{n\to\infty} \inf f_n \right) d\mu \neq \lim_{n\to\infty} \inf \int_X f_n \, d\mu$.

(C) $\int_X \left( \lim_{n\to\infty} \inf f_n \right) d\mu = \lim_{n\to\infty} \sup \int_X f_n \, d\mu$.

(D) $\int_X \left( \lim_{n\to\infty} \inf f_n \right) d\mu \leq \lim_{n\to\infty} \inf \int_X f_n \, d\mu$.

7. Let $\mu$ is a positive measure on $\sigma$–algebra $\mathfrak{M}$. For $E \in \mathfrak{M}$ if $\mu(E) = 0$, then :

(A) $\int_E f d\mu = 0$, even if $f(x) = \infty$ for every $x \in E$.

(B) $\int_E f d\mu$ need not be zero.

(C) $\int_E f d\mu = 0$, only for $f(x) \neq \infty$ for every $x \in E$.

(D) $\int_E f d\mu > 0$.

Turn over

8. Which of the following are false ?

(A) Compact subsets of Hausdorff spaces are closed.

(B) Let F is closed and K is a compact set in a Hausdorff space then $F \cap K$ is compact.

(C) Let F is compact and K is closed in a topological space with $K \subset F$ then K is compact.

(D) Closed subsets of Hausdorff spaces are compact.

9. Let $f$ and $g$ be real (or extended real) functions on a topological space. If $\{x : f(x) > \alpha\}$ and $\{x : g(x) < \alpha\}$ are open for every real $\alpha$, then :

(A) $f$ is upper semicontinuous and $g$ is upper semicontinuous.

(B) $f$ is upper semicontinuous and $g$ is lower semicontinuous.

(C) $f$ is lower semicontinuous and $g$ is upper semicontinuous.

(D) $f$ is lower semicontinuous and $g$ is lower semicontinuous.

10. I : Every set of positive measure has a non measurable subset.

II : Let E be the cantor set, $E \subset \mathbb{R}^1$ then $m$ (E) = 0, where $m$ is the Lebesgue measure.

(A) Both I and II are true.

(B) Both I and II are false.

(C) I is true and II is false.

(D) I is false and II is true.

11. Let $\mathfrak{M}$ be a $\sigma$–algebra in a set X and $E \in \mathfrak{M}$, then :

(A) $|\mu|(E) \neq |\mu(E)|$.

(B) $|\mu|(E) = |\mu(E)|$.

(C) $|\mu|(E) \geq |\mu(E)|$.

(D) $|\mu|(E) \leq |\mu(E)|$.

12. Let X be a set, $\mathfrak{M}$ be a $\sigma$–algebra on X and $\mu$ be a real measure on $\mathfrak{M}$. For every partition $\{E_i\}$ of any set $E \in \mathfrak{M}$, define $|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|$. Then choose the incorrect statement.

    (A) $\frac{|\mu| + \mu}{2}$ is a positive measure on $\mathfrak{M}$.

    (B) $\frac{|\mu| - \mu}{2}$ need not be a positive measure on $\mathfrak{M}$.

    (C) $\frac{|\mu| + \mu}{2}$ is called positive variation of $\mu$.

    (D) $\frac{|\mu| - \mu}{2}$ is called negative variation of $\mu$.

13. Let $\lambda_1, \lambda_2$ are two measures on a $\sigma$-algebra $\mathfrak{M}$ and A, B be two disjoint sets in $\mathfrak{M}$. Also given that $\lambda_1$ is concentrated on A and $\lambda_2$ is concentrated on B. Then

    I : $\lambda_1$ and $\lambda_2$ are mutually singular

    II : $|\lambda_1|$ and $|\lambda_2|$ are mutually singular.

    (A) Both I and II are true.

    (B) Both I and II are false.

    (C) I is true and II is false.

    (D) I is false and II is true.

14. If $\mu = \lambda_1 - \lambda_2$, where $\lambda_1$ and $\lambda_2$ are positive measures, then which of the following is necessarily true ?

    (A) $\lambda_1 = \mu^+$ and $\lambda_2 = \mu^-$.

    (B) $\lambda_1 \neq \mu^+$ and $\lambda_2 \leq \mu^-$.

    (C) $\lambda_1 \leq \mu^+$ and $\lambda_2 \leq \mu^-$.

    (D) $\lambda_1 \geq \mu^+$ and $\lambda_2 \geq \mu^-$.

15. Let $\mu$ be a positive measure, suppose $1 \leq p \leq \infty$, and let $q$ be the exponent conjugate to $p$. Also let $g \in L^q(\mu)$.

I : $\int_X fg \, d\mu$ is a bounded linear functional on $L^p(\mu)$.

II : $\left\| \int_X fg \, d\mu \right\| \leq \|g\|_q$.

Then

(A) Both I and II are true. (B) Both I and II are false.

(C) I is true and II is false. (D) I is false and II is true.

16. Let X be a locally compact Hausdorff space :

(I) Then every bounded linear functional $\phi$ on $C_0(X)$ is represented by a uniqe regular complex borel measure $\mu$.

(II) Norm of every bounded linear functional $\phi$ on $C_0(X)$ is $|\mu|(X)$.

(A) Both I and II are true. (B) Both I and II are false.

(C) I is true and II is false. (D) I is false and II is true.

17. Suppose that $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then $L^q(\mu)$ is the dual space of $L^q(\mu)$.

(A) Only when $\mu$ is $\sigma$–finite. (B) Only when $\mu$ is not $\sigma$–finite.

(C) Even when $\mu$ is not $\sigma$–finite. (D) None of these.

18. Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ are complete measure spaces. Suppose that there exists $A \in \mathscr{S}$, $A \neq \{0\}$, with $\mu(A) = 0$, $B \subset Y$, $B \notin \mathscr{T}$. Then :

(A) $(\mu \times \lambda)(A \times Y) = 0$ but $A \times B \notin \mathscr{S} \times \mathscr{T}$.

(B) $(\mu \times \lambda)(A \times Y) = 0$ but $A \times B \in \mathscr{S} \times \mathscr{T}$.

(C) $(\mu \times \lambda)(A \times Y) \neq 0$ but $A \times B \in \mathscr{S} \times \mathscr{T}$.

(D) $(\mu \times \lambda)(A \times Y) \neq 0$ but $A \times B \notin \mathscr{S} \times \mathscr{T}$.

19. Let $\mu = \lambda = m$, be the lebesgue measure on $\mathbb{R}^1$. Also let A is a singleton set, B be an $y$ non-measurable set in $\mathbb{R}^1$ and $m_2$ be the lebesgue measure in $\mathbb{R}^2$.

(A) $m_1 \times m_1$ is not a complete measure and $m_2$ is the completion of $m_1 \times m_1$.

(B) $m_1 \times m_1$is not a complete measure and $m_2$ is not the completion of $m_1 \times m_1$.

(C) $m_1 \times m_1$ is a complete measure and $m_2$ is not the completion of $m_1 \times m_1$.

(D) $m_1 \times m_1$ is a complete measure and $m_2 = m_1 \times m_1$.

20. Let $m_n$ denote the lebesgue measure in $\mathbb{R}^n$ for natural number $n$.

I : $m_5$ is not a completion of $m_1 \times m_4$.

II : $m_5$ is not a completion of $m_2 \times m_3$.

(A) Both I and II are true.

(B) Both I and II are false.

(C) I is true and II is false.

(D) I is false and II is true.