D 140688 **(Pages : 3)** **Name..................................**

Reg. No...............................

SECOND SEMESTER M.Sc. (CBCSS) REGULAR/SUPPLEMENTARY DEGREE EXAMINATION, APRIL 2026

Mathematics

MTH 2C 07—REAL ANALYSIS—II

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question carries a weightage of 1.

1. Verify whether the set $\mathbb{N}$ of natural numbers is a Borel set.
2. Find $m(\mathbb{Q})$ where m is the Lebesgue measure and $\mathbb{Q}$ is the set of all rationals.
3. Let $f(x)$ be defined on the reals by

$$f(x)=\begin{cases}1 & \text{if } x \text{ is an integer}\\ 0 & \text{otherwise.}\end{cases}$$

   Verify whether f is Lebesgue measurable.
4. Let

$$f(x)=\begin{cases}1 & \text{if } 0\le x<1\\ 0 & \text{if } 1\le x\le 2.\end{cases}$$

   Find $\int_{[0,2]} f$.
5. Let $f(x)$ in [0, 1] be defined by

$$f(x)=\begin{cases}0 & \text{if } x \text{ is rational}\\ 1 & \text{otherwise.}\end{cases}$$

   Find $\int_{E} f$ where E is the set of all rationals in [0, 1].
6. Let $F=\{f_1, f_2, \ldots, f_n\}$ be a family of integrable functions on [0, 1]. Verify whether F is uniformly integrable in [0, 1].
7. Show that every increasing function on [0, 1] is of bounded variation.
8. Let $f(x) = x^2$ in [0, 1]. Show that $f(x)$ is absolutely continuous.

(8 × 1 = 8 weightage)

Turn Over

Part B

Answer any **two** *questions from each module.*
Each question carries a weightage of 2.

Module I

9. Let $\mathbb{N}$ be the set of all natural numbers. Show that $m^*(\mathbb{N}) = 0$ where m^* is the Lebesgue outer measure.
10. Show that if A and B are measurable sets then $A \cup B$ is measurable.
11. Show that if (B_n) is a descending chain of Lebesgue measurable sets of finite measure then $m(\cap_n B_n) = \lim_{n\to\infty} m(B_n)$.

Module II

12. Let E be a measurable set of finite measure and ϕ, ψ be a simple functions on E. Show that $\int_E(\phi - \psi) = \int_E \phi - \int_E \psi$.
13. Let f be a bounded measurable function on a set E of finite measure. Show that $\left|\int_E f\right| \le \int_E |f|$.
14. Let E be a measurable set. Let (f_n) be a sequence of non-negative integrable functions on E such that $\lim_{n\to\infty}\int_E f_n = 0$. Show that (f_n) converges to 0 in measure.

Module III

15. Let f be a function of bounded variation on [0, 1]. Show that f is differentiable a.e. on (0, 1).
16. Let f be absolutely continuous on [0, 1]. Show that f is of bounded variation on [0, 1].
17. Show that every rapidly Cauchy sequence in a normed linear space X is a Cauchy sequence in X.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (a) Define Lebesgue measurable set.
    (b) Show that a subset E of $\mathbb{R}$ is measurable if and only if for each $\varepsilon > 0$ there exists an open set O containing E such that $m^*(O \sim E) < \varepsilon$ where m^* is the Lebesgue outer measure on $\mathbb{R}$.
19. (a) Let (f_n) be a sequence of measurable functions on a measurable set E such that $f_n \to f$ pointwise on E. Show that f is measurable.
    (b) Show that the characteristic function χ_A is measurable if and only if A is measurable.

20. (a) Define uniformly integrable family of functions.

(b) Let (f_n) be a sequence of functions which is uniformly integrable over a set E of finite measure. Show that if $f_n \to f$ pointwise a.e. on E then f is integrable on E.

21. (a) Let f be a monotone function on an open interval (a, b). Show that f is continuous except possibly at a countable number of points in (a, b).

(b) Let f be an increasing function on $[a, b]$. Show that f' is integrable over $[a, b]$ and $\int_a^b f' \leq f(b) - f(a)$.

(2 × 5 = 10 weightage)