D 143206 **(Pages : 3)** **Name..................................**

Reg. No...............................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2026

(CBCSS)

Mathematics

MTH4E08—COMMUTATIVE ALGEBRA

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*

Each question has weightage 1.

1. Let A be a ring and a. be an ideal of A. Verify whether $f : A \to A/a$ defined by $f(x) = x + a$ is a ring homomorphism.
2. Verify whether the ideal generated by 10 is maximal in the ring $\mathbb{Z}$ of integers.
3. Let a = (3) be the ideal generated by 3 in the ring $\mathbb{Z}$ of integers. Find an ideal b such that $a + b = (1)$.
4. Let p be a prime ideal of a ring A. Verify whether S = A – p is a multiplicatively closed suset of A.
5. Let A = $\mathbb{Z}$ and S = A – p where p is the ideal generated by 5. Verify whether $2/3 \in S^{-1}A$.
6. Let M be an A-module and S be a multiplicatively closed subset of A. Describe the module $S^{-1}M$.
7. Let B = $k[x, y]$ and A = $k[x]$ be polynomial rings over a field k. Verify whether y is integral over A.
8. Verify whether the polynomial ring $k[x]$ where k is a field satisfies a.c.c. on ideals.

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any **two** *questions from each module.*
Each question has weightage 2.

Module I

9. Let $f : \mathrm{A} \to \mathrm{B}$ be a ring homomorphism from A onto B. Show that A/*ker f* is isomorphic to B.

10. Let m be an ideal of a ring A such that for all $x \in \mathrm{A} - m, x$ is a unit. Show that m is a maximal ideal of A.

11. Let $\mathcal{R}$ be the Jacobson radical of a ring A. Show that if $x \in \mathcal{R}$ then $1 - xy$ is a unit in A for every $y \in \mathrm{A}$.

Module II

12. Let S be a multiplicatively closed subset of a ring A and let $f : \mathrm{A} \to \mathrm{S}^{-1}\mathrm{A}$ be given by $f(a) = a/1$. Show that every element of $\mathrm{S}^{-1}\mathrm{A}$ is of the form $f(a)\,(f(s))^{-1}$ for some $a \in \mathrm{A}$ and $s \in \mathrm{S}$.

13. Let N, P be submodules of an A-module M. Show that $\mathrm{S}^{-1}(\mathrm{N} \cap \mathrm{P}) = \mathrm{S}^{-1}\mathrm{N} \cap \mathrm{S}^{-1}\mathrm{P}$.

14. Let $\phi : \mathrm{M} \to \mathrm{N}$ be an A-module homomorphism. Show that if (ϕ) is injective then for all prime ideals p of A the homomorphism $\phi_p : \mathrm{M}_p \to \mathrm{N}_p$ is also injective.

Module III

15. Let A be a subring of a ring B. Show that

$$\mathrm{C} = \{x \in \mathrm{B} : x \text{ is integral over A}\}$$

is a subring of B.

16. Let $0 \to \mathrm{M}' \xrightarrow{\alpha} \mathrm{M} \xrightarrow{\beta} \mathrm{M}'' \to 0$ be an exact sequence of A-modules. Show that if M is Noetherian then M' and M" are Noetherian.

17. Let S be a multiplicative subset of a ring A. Show that if A is Noetherian then $\mathrm{S}^{-1}\mathrm{A}$ is Noetherian.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*

Each question has weightage 5.

18. (a) Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of A-modules. Show that

    (i) f is injective,

    (ii) g is surjective.

    (b) Show that a sequence of A-modules $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ is exact if and only if for every A-module N the sequence

$$0 \to \mathrm{H}om(M'', N) \xrightarrow{v^*} \mathrm{H}om(M, N) \xrightarrow{u^*} \mathrm{H}om(M', N)$$

    is exact.

19. (a) Define primary ideal of a ring A.

    (b) Let q be a primary ideal of a ring A. Show that $r(q)$ is the smallest prime ideal containing q.

    (c) Let A = $k[x, y]$ where k is a field and q be the ideal generated by $\{x, y^2\}$. Show that q is a primary ideal of A.

20. (a) Let $a = \cap q_i$ be a minimal primary decomposition of an ideal a. Let $p_i = r(qi)$ for each i. Show that for each i, $p_i = r(a : x)$ for some $x \in A$.

    (b) Let S be a multiplicatively closed subset of a ring A and q be a p-primary ideal of A. Show that if $S \cap p \neq \emptyset$ then $S^{-1}q = S^{-1}A$.

21. Let M be an A- module. Show that

    (a) M is Noetherian if and only if every submodule of M is finitely generated.

    (b) If M satisfies both a.c.c and d.c.c. on submodules then M has a composition series.

(2 × 5 = 10 weightage)