D 150349 **(Pages : 4)** **Name.....**

Reg. No.....

THIRD SEMESTER M.Sc. (CBCSS) [REGULAR/SUPPLEMENTARY] DEGREE EXAMINATION, NOVEMBER 2026

Mathematics

MTH 3E 03—MEASURE AND INTEGRATION

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A (Short Answer Type Questions)

Answer **all** *questions.*
Each question carries a weightage 1.

1. Define topology on a set. Give an example.
2. Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then prove that $A \subset B$ implies $\mu(A) \leq \mu(B)$ $A \in \mathfrak{M}$, $B \in \mathfrak{M}$.
3. Prove that compact subsets of Hausdorff spaces are closed.
4. Suppose μ, λ_1, and λ_2 are measures on a σ-algebra $\mathfrak{M}$, and μ is positive. If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then prove that $\lambda_1 + \lambda_2 \perp \mu$.
5. Define positive and negative variations of a measure.
6. Define monotone class $\mathfrak{M}$.
7. If μ is a positive or real Borel measure on R^k, then prove that $\bar{D}\mu$ is a Borel function.
8. If $\mu \perp m$, then prove that $(D\mu)(x) = 0$ a.e. $[m]$.

($8 \times 1 = 8$ weightage)

Part B (Paragraph Type Questions)

Answer any **two** *questions from each module.*
Each question carries a weightage 2.

Module I

9. Let X and Y be topological spaces. Prove that a mapping f of X into Y is continuous if and only if f is continuous at every point of X.

Turn over

10. If $f_n : X \to [0, \infty]$ is measurable, for n = 1, 2, 3, ..., and

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \, (x \in X),$$

then prove that

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

11. Suppose $V_1, ..., V_n$ are open subsets of a locally compact Hausdorff space X, K is compact, and

$$K \subset V_1 \cup ..... \cup V_n.$$

Then prove that there exist functions $h_i \prec V_i \; (i = 1, ..., n)$ such that

$$h_1(x) + ... + h_n(x) = 1 \, (x \in K).$$

Module II

12. Suppose $f \in L^1(\mu)$, is real-valued, and $\varepsilon > 0$. Then prove that there exist functions u and v on X such that $u \leq f \leq v$, u is upper semicontinuous and bounded above, v is lower semicontinuous and bounded below, and

$$\int_X (v - u) \, d\mu < \varepsilon.$$

13. Prove that the total variation $|\mu|$ of a complex measure μ on $\mathfrak{M}$ is a positive measure on $\mathfrak{M}$.

14. If μ is a complex measure on X, then prove that

$$|\mu|(X) < \infty.$$

Module III

15. Let μ be a real measure on a σ-algebra $\mathfrak{M}$ in a set X. Then prove that there exist sets A and $B \in \mathfrak{M}$ such that $A \cup B = X$, $A \cap B = \varnothing$, and such that the positive and negative variations μ^+ and μ^- of μ satisfy

$$\mu^+(E) = \mu(A \cap E), \; \mu^-(E) = -\mu(B \cap E) \, (E \in \mathfrak{M}).$$

16. Suppose μ is a complex Borel measure on R^1 and

$$f(x) = \mu\left((-\infty, x)\right)\left(x \in R^1\right).$$

Prove that, for any $x \in R^1$, each of the following two statements implies the other :

(a) f is differentiable at x and $f'(x) = A$.

(b) To every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{\mu(I)}{m(I)} - A\right| < \varepsilon.$$

for every open segment I which contains x and whose length is less than δ.

17. Suppose μ and λ are real Borel measures on R^k, $\sigma = \mu + \lambda$, and $x \in R^k$. Then prove that

$$\left(\bar{D}\sigma\right)(x) \leq \left(\bar{D}\mu\right)(x) + \left(\bar{D}\lambda\right)(x)$$

unless the right side is undefined. If μ and λ are differentiable at x, so is σ, and

$$\left(D\sigma\right)(x) = \left(D\mu\right)(x) + \left(D\lambda\right)(x).$$

(6 × 2 = 12 weightage)

Part C (Essay Type Questions)

Answer **two** *questions.*
Each question carries a weightage 5.

18. Let $\{f_n\}$ be a sequence of measurable functions on X and suppose that

(a) $0 \leq f_1(x) \leq f_2(x) \leq \ldots \leq \infty$ for every $x \in X$.

(b) $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$.

Then prove that f is measurable, and

$$\int_X f_n \, d\mu \to \int_X f \, d\mu \text{ as } n \to \infty.$$

Turn over

19. Suppose X is a locally compact, σ-compact Hausdorff space. Then prove that $\mathfrak{M}$ and μ have the following properties :

    (a) If $E \in \mathfrak{M}$ and $\varepsilon > 0$, there is a closed set F and an open set V such that $F \subset E \subset V$ and $\mu(V - F) < \varepsilon$.

    (b) μ is a regular Borel measure on X.

    (c) If $E \in \mathfrak{M}$ there are sets A and B such that A is an F_σ, B is a G_δ, $A \subset E \subset B$, and $\mu(B - A) = 0$.

20. Suppose $1 \leq p < \infty$, μ is a σ-finite positive measure on X, and Φ is a bounded linear functional on $L^p(\mu)$. Then prove that there is a unique $g \in L^q(\mu)$, where q is the exponent conjugate to p, such that

$$\Phi(f) = \int_X fg \, d\mu \quad \left(f \in L^p(\mu)\right).$$

21. Suppose μ is a real Borel measure on R^1, $\mu \geq 0$, and $\mu \perp m$. Let Ω be the family of all open segments in R^1. If

$$S = \left\{x : \left(\bar{D}\mu\right)(x) > 0\right\} \cap \left\{x : \left(\underline{D}\mu\right)(x) < \infty\right\},$$

then prove that $m(S) = \mu(S) = 0$.

(2 × 5 = 10 weightage)

D 150349-A **(Pages : 6)** **Name.....**

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Mathematics

MTH 3E 03—MEASURE AND INTEGRATION

(2019 Admission onwards)

(Multiple Choice Questions for SDE Candidates)

Time : 20 Minutes	**Total No. of Questions : 20**	**Maximum : 5 Weightage**

INSTRUCTIONS TO THE CANDIDATE

1. This Question Paper carries Multiple Choice Questions from 1 to 20.
2. The candidate should check that the question paper supplied to him/her contains all the 20 questions in serial order.
3. Each question is provided with choices (A), (B), (C) and (D) having one correct answer. Choose the correct answer and enter it in the main answer-book.
4. The MCQ question paper will be supplied after the completion of the descriptive examination.

MTH 3E 03—MEASURE AND INTEGRATION

(Multiple Choice Questions for SDE Candidates)

1. Let $(\mathbb{R}, \tau)$ be a topological space and given that {1, 2, 3} and {2, 4, 6, 8, ...} are open sets in the given topology $(\mathbb{R}, \tau)$. Then which of the following is necessarily true ?

   (A) {1, 3, 4} is an open set in $(\mathbb{R}, \tau)$. (B) {2, 4, 6, 8, ...} is a closed set in $(\mathbb{R}, \tau)$.

   (C) {2} is an open set in $(\mathbb{R}, \tau)$. (D) {0, 1, 2, 4, 6, ...} is an open set in $(\mathbb{R}, \tau)$.

2. Let X be a non-empty set (X, τ_1) be the discrete topological space, (X, τ_2) be a topological space, then :

   (A) There exists no continuous map $f : X \to X$.

   (B) There exists continuous map $f : X \to X$ depending on τ_2.

   (C) There exists exactly one continuous map $f : X \to X$.

   (D) Any map $f : X \to X$ is continuous.

3. Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then

   (A) $\mu(\phi) \neq 0$.

   (B) $\mu(A_1 \cup A_2 \cup \ldots \cup A_n) < \mu(A_1) + \ldots + \mu(A_n)$ if A_1, A_2, ..., A_n are pairwise disjoint members of $\mathfrak{M}$.

   (C) $A \subset B$ implies $\mu(A) \geq \mu(B)$ if $A, B \in \mathfrak{M}$.

   (D) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$, and $A_1 \subset A_2 \subset \ldots$.

4. Let $\mathbb{Q}$ be the set of all rational numbers then Lebesgue measure of $\mathbb{Q} \cap [0, 1]$ is :

   (A) 0. (B) 1.

   (C) 2. (D) ∞.

5. Let $A, B \in \mathfrak{M}$ and $A \subset B$, also given that $\mu(B) = 1$. Then which of the following are necessarily true ?

   (A) $\mu(A) = 1$. (B) $\mu(A) > 1$.

   (C) $0 \leq \mu(A) < 1$. (D) $0 \leq \mu(A) \leq 1$.

6. If $f_n : X \to [0, \infty]$ is measurable, for each positive integer n and μ is a positive measure on σ-algebra $\mathfrak{M}$ then :

   (A) $\int_X \left(\lim_{n\to\infty} \inf f_n \right) d\mu = \lim_{n\to\infty} \inf \int_X f_n d\mu.$

   (B) $\int_X \left(\lim_{n\to\infty} \inf f_n \right) d\mu \neq \lim_{n\to\infty} \inf \int_X f_n d\mu.$

   (C) $\int_X \left(\lim_{n\to\infty} \inf f_n \right) d\mu = \lim_{n\to\infty} \sup \int_X f_n d\mu.$

   (D) $\int_X \left(\lim_{n\to\infty} \inf f_n \right) d\mu \leq \lim_{n\to\infty} \inf \int_X f_n d\mu.$

7. Let μ is a positive measure on σ-algebra $\mathfrak{M}$. For $E \in \mathfrak{M}$ if $\mu(E) = 0$, then :

   (A) $\int_E f d\mu = 0,$ even if $f(x) = \infty$ for every $x \in E.$

   (B) $\int_E f d\mu$ need not be zero.

   (C) $\int_E f d\mu = 0,$ only for $f(x) \neq \infty$ for every $x \in E.$

   (D) $\int_E f d\mu > 0.$

8. Choose the false statements :

   (A) A set E in a measure space (with measure μ) is said to have σ-finite measure if E is a countable union of set E_i, with $\mu(E_i) < \infty.$

   (B) A set E in a topological space is called σ-compact if E is a union of compact sets.

   (C) Every σ-compact set has σ-finite measure.

   (D) If $E \in \mathfrak{M}$ and E has σ-finite measure, then E is inner regular.

9. If f is a continuous complex function on $[a, b]$, then :

   (A) Riemann integral of f is less than the Lebesgue integral of f over $[a, b]$.

   (B) Riemann integral of f is greater than the Lebesgue integral of f over $[a, b]$.

   (C) Riemann integral of f is equal to the Lebesgue integral of f over $[a, b]$.

   (D) Riemann integral of f and Lebesgue integral of f over $[a, b]$ are not comparable.

Turn over

10. If $A \subset \mathbb{R}^1$ and every subset of A is Lebesgue measurable then :

(A) $m(A) = \infty$. (B) $m(A) = 0$.

(C) $0 < m(A) < \infty$. (D) $m(A) > 1$.

11. I. Every set of positive measure has a non-measurable subset.

II. Let E be the cantor set, $E \subset \mathbb{R}^1$ then $m(E) = 0$, where m is the Lebesgue measure.

(A) Both I and II are true. (B) Both I and II are false.

(C) I is true and II is false. (D) I is false and II is true.

12. Let $\mathfrak{M}$ be a σ-algebra in a set X and $E \in \mathfrak{M}$, then :

(A) $|\mu|(E) \neq |\mu(E)|$. (B) $|\mu|(E) = |\mu(E)|$.

(C) $|\mu|(E) \geq |\mu(E)|$. (D) $|\mu|(E) \leq |\mu(E)|$.

13. Let μ is a complex measure on X, then choose the false statement :

(A) $|\mu|$ is countably additive.

(B) $|\mu|(X) < \infty$.

(C) $|\mu|(\phi) \neq 0$.

(D) Collection of all complex measure on σ-algebra, $\mathfrak{M}$ is a vector space.

14. Let μ, λ be two complex measure on σ-algebra, $\mathfrak{M}$ and c be any scalar. Define $(\mu + \lambda)(E) = \mu(E) + \lambda(E)$ and $(c\mu)(E) = c\,(\mu(E))$ for $E \in \mathfrak{M}$. Then choose the correct statement from below :

(A) $\mu - \lambda$ is also a complex measure.

(B) $c\mu$ is a complex measure only for $c = 0$.

(C) there exists some complex measure λ for which $\mu + \lambda$ is not a complex measure.

(D) $\mu + c\lambda$ is a complex measure but $c\mu + \lambda$ need not be a complex measure.

15. Let X be a set, $\mathfrak{M}$ be a σ-algebra on X and define $\|\mu\| = |\mu|(X)$ for $\mu \in \mathfrak{M}$. Then which of the following is necessarily true ?

(A) $\|\mu + \lambda\| = \|\mu\| + \|\lambda\|$. (B) $\|\mu + \lambda\| \leq \|\mu\| + \|\lambda\|$.

(C) $\|\mu + \lambda\| > \|\mu\| + \|\lambda\|$. (D) $\|\mu + \lambda\| < \|\mu\| + \|\lambda\|$.

16. Let μ be a positive measure, suppose $1 \le p \le \infty$, and let q be the exponent conjugate to p. If $g \in L^q(\mu)$ and $\phi_g(f) = \int_X fg \, d\mu$ then choose incorrect statement.

(A) ϕ_g is bounded linear functional on $L^p(\mu)$.

(B) ϕ_g has norm atmost $\|g\|_q$.

(C) For $p = \infty$ all bounded linear functionals on $L^p(\mu)$ have the form $\phi_g(f) = \int_X fg \, d\mu$.

(D) For $1 < p < \infty$ all bounded linear functionals on $L^p(\mu)$ have the form $\phi_g(f) = \int_X fg \, d\mu$.

17. Let X be a locally compact Hausdorff space. Choose the incorrect statement :

(A) $C_c(X)$ is a dense subspace of $C_o(X)$, relative to supnorm.

(B) Bounded linear functional on $C_c(X)$ has an extension to a bounded linear function on $C_o(X)$.

(C) Bounded linear functional on $C_c(X)$ has more than two extension to a bounded linear function on $C_o(X)$.

(D) $C_o(X)$ is a banach space.

18. Let f be an $(S \times I)$-measurable function on $X \times Y$, then for all $x \in X$ and $y \in Y$:

(A) f_x is I-measurable and f^y is I-measurable.

(B) f_x is I-measurable and f^y is S-measurable.

(C) f_x is S-measurable and f^y is I-measurable.

(D) f_x is S-measurable and f^y is S-measurable.

19. Let (X, S, μ) and (Y, I, λ) are σ-finite measure space. Suppose that $Q \in S \times I$ and define $\phi(x) = \lambda(Q_x), \psi(y) = \mu(Q^y)$ for all $x \in X, y \in Y$. Then :

(A) ϕ is I-measurable and ψ is I-measurable.

(B) ϕ is I-measurable and ψ is S-measurable.

(C) ϕ is S-measurable and ψ is I-measurable.

(D) ϕ is S-measurable and ψ is S-measurable.

Turn over

20. Let (X, S, μ) and (Y, I, λ) are σ-finite measure space. Suppose that $Q \in S \times I$ and define $(\mu \times \lambda)(Q) = \int_X \lambda(Q_x) d\mu(x) = \int_X \mu(Q^y) d\lambda(y)$.

I : μ and λ is a σ-finite.

II : μ and λ is countable subadditive not countable additive on $S \times I$.

(A) Both I and II are true.

(B) Both I and II are false.

(C) I is true and II is false.

(D) I is false and II is true.