D 150345 **(Pages : 3)** **Name.....**

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE (CBCSS) REGULAR / SUPPLEMENTARY EXAMINATION, NOVEMBER 2026

Mathematics

MTH3C13—FUNCTIONAL ANALYSIS

(2019 Admission Onwards)

Time : Three Hours Maximum : 30 Weightage

Part A – Short Answer Type Questions

Answer **all** *questions.*

Each question carries a weightage 1.

1. Define Banach space. Give an example.
2. State Minkowski's inequality.
3. Define innerproduct. Give an example.
4. Prove that in a Hilbert space H,

   $$\|x+y\|^2+\|x-y\|^2=2\left(\|x\|^2+\|y\|^2\right)$$

   for all $x, y \in \mathrm{H}$.
5. If $\{f_i\}$ is a complete system in a Hilbert space H and $x \perp f_i$ then prove that $x = 0$.
6. If f is a bounded functional, then prove that f is a continuous functional.
7. Let $\mathrm{L} \mapsto \mathrm{E}$ and codim L = 1. Then, prove that there exists $f \in \mathrm{E}^{\#}$ such that $\ker f = \mathrm{L}$.
8. Prove that for all $x_1 \in \mathrm{X}$ and for all $x_2 \in \mathrm{X}$ such that $x_1 \neq x_2$, there exists $f \in \mathrm{X}^*$ satisfying $f(x_1) \neq f(x_2)$.

(8 × 1 = 8 weightage)

Turn over

Part B - Paragraph Type Questions

Answer any **two** *questions from each module.*
Each question carries a weightage 2.

Module I

9. Prove that for every sequence of scalars (a_i) and (b_i) and for $1 < p < \infty$

$$\left|\sum a_k b_k\right| \le \sum |a_k b_k| \le \left(\sum |a_k|^p\right)^{1/p} \left(\sum |b_k|^q\right)^{1/q}$$

where $1/p + 1/q = 1$.

10. Prove that the space $l_p, 1 \le p < \infty,$ is a complete space.

11. For every sequence of scalars $a = (a_i)$ and $b = (b_i)$ and for $1 \le p \le \infty$, prove that :

$\|a + b\|_p \le \|a\|_p + \|b\|_p$.

Module II

12. State and prove Cauchy-Shwartz inequality for a linear space with an inner product.

13. If $\{e_i\}_1^n$ are pairwise orthogonal and normalized in H, then prove that $\left\|\sum_1^n \alpha_i e_i\right\| = \left(\sum_1^n |\alpha_i|^2\right)^{1/2}$.

14. Prove that the Hilbert space H is separable if and only if there exists a complete orthonormal system $\{e_i\}_{i \ge 1}$.

Module III

15. Let $\mathrm{L} \mapsto \mathrm{X}$ be a subspace of a normed space X and let $x \in \mathrm{X}$ such that $\mathrm{dist}(x, \mathrm{L}) = d > 0$. Then, prove that there exists $f \in \mathrm{X}^*$ such that $\|f\| = 1, f(\mathrm{L}) = 0$ and $f(x) = d$.

16. Let $\{x_1, \ldots, x_n\} \subseteq \mathrm{X}$ be a linearly independent subset of X. Then, prove that there exists $f_1, \ldots, f_n \in \mathrm{X}^*$ such that $f_i(x_j) = \delta_{ij}$ and every $x \in \mathrm{span}\{x_i\}_1^n$ is decomposed as $x = \sum_1^n f_i(x) x_i$.

17. If $\mathrm{A} \in \mathrm{K}(\mathrm{X} \mapsto \mathrm{Y}), \mathrm{B} \in \mathrm{L}(\mathrm{Z} \mapsto \mathrm{X}), \mathrm{C} \in \mathrm{L}(\mathrm{Y} \mapsto \mathrm{Z})$, then prove that $\mathrm{AB} \in \mathrm{K}(\mathrm{Z} \mapsto \mathrm{Y})$, and $\mathrm{CA} \in \mathrm{K}(\mathrm{X} \mapsto \mathrm{Z})$.

(6 × 2 = 12 weightage)

Part C - Essay Type Questions.

Answer **two** *questions.*

Each question carries a weightage 5.

18. Let E, X_i for $i = 1, 2$ be linear normed spaces with X_i being complete and $T_i : E \mapsto X_i$ isometries into dense subspaces of X_i for $i = 1,2$. Then prove that the natural isometry $T_2 \circ T_1^{-1} : T_1 E \mapsto T_2 E$ can be extended to an isometry between X_1 and X_2.

19. Prove that the system $\left\{\left(1/\sqrt{2\pi}\right)e^{\text{int}}\right\}_{-\infty}^{\infty}$ is an orthonormal basis of $L_2[-\pi, \pi]$.

20. State and prove Riesz representation theorem.

21. Let X be a normed space and let Y be a complete normed space. Then prove that $L(X \mapsto Y)$ is a Banach space.

(2 × 5 = 10 weightage)