D 150343 **(Pages : 3)** **Name.....**

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2026

(CBCSS)

Mathematics

MTH 3C 11—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A (Short Answer Type Questions)

Answer **all** *questions.*
Each question carries a weightage 1.

1. If I is the identity operator on R^n, then prove that $\det[I] = \det(e_1, ..., e_n) = 1$.
2. Suppose X is a vector space, and dim X = n. Prove that a set E of n vectors in X spans X if and only if E is independent.
3. Prove that $A0 = 0$, if A is linear.
4. Find the signed curvature of the catenary
5. Prove that the total signed curvature of a closed plane curve is an integer multiple of 2π.
6. If $\mathcal{S}$ is a surface and $p \in \mathcal{S}$, prove that the derivative at p of the identity map $\mathcal{S} \to \mathcal{S}$ is the identity map $T_p\mathcal{S} \to T_p\mathcal{S}$.
7. Find ∇f and $\| \nabla f \|$ of $f(x, y, z) = x^2 + y^2 - z^2$.
8. Define the Gaussian curvature and the mean curvature.

(8 × 1 = 8 weightage)

Turn over

Part B (Paragraph Type Questions)

Answer any **two** *questions from each module*
Each question carries a weightage 2.

MODULE I

9. Suppose f maps a convex open set $E \subset R^n$ into R^m, f is differentiable in E, and there is a real number M such that

$$\| f'(x) \| \leq M$$

for every $x \in E$. Then prove that

$$| f(b) - f(a) | \leq M | b - a |$$

for all $a \in E, b \in E$.

10. Suppose f is a C'-mapping of an open set $E \subset R^n$ into R^n, $f'(a)$ is invertible for some $a \in E, b = f(a)$. Then prove that there exist open sets U and V in R^n such that $a \in U, b \in V, f$ is one-to-one on U, and $f(U) = V$.

11. Suppose $E = \{e_1, ..., e_n\}$ and $U = \{u_1, ..., u_n\}$ are bases in R^n. Then prove that for any linear operator A on R^n,

$$\det [A]_U = \det [A]_E .$$

MODULE II

12. Prove that a parametrized curve has a unit-speed reparametrization if and only if it is regular.

13. Let $k : (\alpha, \beta) \to \mathbb{R}$ be any smooth function. Then prove that there is a unit-speed curve $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ whose signed curvature is k.

14. Let S and $\tilde{S}$ be surfaces and let $f : S \to \tilde{S}$ be a smooth map. Then, prove that f is a local diffeomorphism if and only if, for all $p \in S$, the linear map $D_p f : T_p S \to T_{f(p)} \tilde{S}$ is invertible.

MODULE III

15. Prove that every compact surface is orientable.

16. Show that, if $\gamma:(\alpha,\beta)\to\mathbb{R}^3$ is a curve whose image is contained in a surface patch $\sigma:\mathrm{U}\to\mathbb{R}^3$, then $\gamma(t)=\sigma\big(u(t),v(t)\big)$ for some smooth map $(\alpha,\beta)\to\mathrm{U}, t\mapsto\big(u(t),v(t)\big)$.

17. Let $\mathcal{S}$ be a (connected) surface of which every point is an umbilic. Then prove that $\mathcal{S}$ is an open subset of a plane or a sphere.

(6 × 2 = 12 weightage)

Part C (Essay Type Questions)

Answer **two** *questions.*
Each question carries a weightage 5.

18. State and prove the implicit function theorem.

19. Let $\gamma(s)$ and $\tilde{\gamma}(s)$ be two unit-speed curves in $\mathbb{R}^3$ with the same curvature $\kappa(s)>0$ and the same torsion $\tau(s)$ for all s. Then prove the following: There is a direct isometry M of $\mathbb{R}^3$ such that

$$\tilde{\gamma}(s)=\mathrm{M}\big(\gamma(s)\big) \text{ for all } s.$$

Further, if k and t are smooth functions with $k > 0$ everywhere, there is a unit-speed curve in $\mathbb{R}^3$ whose curvature is k and whose torsion is t.

20. Let $\sigma:\mathrm{U}\to\mathbb{R}^3$ be a patch of a surface $\mathcal{S}$ containing a point $p\in\mathcal{S}$, and let (u,v) be co-ordinates in U. Prove that the tangent space to $\mathcal{S}$ at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v (the derivatives are evaluated at the point $(u_0,v_0)\in\mathrm{U}$ such that $\sigma(u_0,v_0)=p$).

21. Prove that a smooth map $f:\mathcal{S}_1\to\mathcal{S}_2$ is a local isometry if and only if the symmetric bilinear forms $\langle\,,\,\rangle_p$ and $f^*\langle\cdot,\cdot\rangle_p$ on $\mathrm{T}_p\mathcal{S}_1$ are equal for all $p\in\mathcal{S}_1$.

(2 × 5 = 10 weightage)