D 52824 **(Pages : 3)** Name.......................................

Reg. No..................................

FIRST SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2023

(CBCSS)

Mathematics

MTH 1C 02—LINEAR ALGEBRA

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question carries a weightage 1.

1. Is the vector $(3,-1,0,-1)$ in the subspace of $\mathbb{R}^4$ spanned by the vectors $(2,-1,3,2),(-1,1,1,-3)$ and $(1,1,9,-5)$.

2. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.

3. Prove or disprove "Every square matrix has characteristic values in $\mathbb{R}$".

4. Define linear functional. Give an example.

5. Let V be an inner product space and let $x \in V$. Prove that if $(x / y) = 0 \ \forall\ y \in V \Rightarrow x = 0$.

6. Prove that the map $f : R^2 \to R^3$ defined by $f(x, y) = (x+1, 2y, x+y)$ is not linear.

7. Define inner product on a vector space V .

8. Let T be a linear operator on V and let U be any linear operator on V which commutes with T, i.e., $TU = UT$. Let W be the range of U and let N be the null space of U. Show that both W and N are invariant under T.

(8 × 1 = 8 weightage)

Turn over

Part B (Paragraph Type Questions)

Answer any **six** *questions, choosing* **two** *questions from each module.*
Each question carries a weightage 2.

MODULE I

9. Show that the subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

10. Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for $\mathbb{R}^3$ consisting of $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 1, 1)$, $\alpha_3 = (1, 0, 0)$. What are the co-ordinates of the vector (a, b, c) in the ordered basis B.

11. Let T be the operator on $\mathbb{C}^2$ for which $T\varepsilon_1 = (1, 0, i), T\varepsilon_2 = (0, 1, 1), T\varepsilon_3 = (i, 1, 0)$. Is T invertible ?

MODULE II

12. Let W_1 and W_2 be subspaces of a finite dimensional vector space V . Prove that $W_1 = W_2$ if and only if $W_1{}^0 = W_2{}^0$.

13. Let $f_1(x_1, x_2, x_3, x_4) = x_1 + 2x_2 + 2x_3 + x_4$; $f_2(x_1, x_2, x_3, x_4) = 2x_2 + x_4$; $f_3(x_1, x_2, x_3, x_4) = -2x_1 - 4x_3 + 3x_4$ be three linear functionals on $\mathbb{R}^4$. Find the subspace which these functionals annihilate .

14. Let T be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$, each vector of matrix which is a characteristic vector of T .

MODULE III

15. Let $E_1, \ldots, E_k$ are k linear operators on V which satisfy :

    (i) each E_i is a projection ;

    (ii) $E_iE_j = 0$ if, $i \neq j$;

(iii) $I = E_1 + ... + E_k$;

(iv) the range of E_i is W_i

and let W_i be the range of E_i, then show that $V = W_1 \oplus .... \oplus W_k$.

16. Let V be a real or complex vector space with an inner product. Prove that

$\| \alpha + \beta \|^2 + \| \alpha - \beta \|^2 = 2 \| \alpha \|^2 + 2 \| \beta \|^2$ for every $\alpha, \beta \in V$.

17. State and prove Bessel's Inequality.

(6 × 2 = 12 weightage)

Part C (Essay Type Questions)

Answer any **two** *questions.*
Each question carries a weightage 5.

18. State and prove Cayley- Hamilton Theorem.

19. a) Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1) ..... (x - c_k)$ where $c_1, ..., c_k$ are distinct elements of F.

    b) Define T-conductor of α into W.

20. Let $g, f_1, ..., f_r$ be linear functionals on a vector space V with respective null spaces $N, N_1, ..., N_r$. Then show that g is a linear combination of $f_1, ..., f_r$ if and only if N contains the intersection $N_1 \cap ... \cap N_r$.

21. State and prove Gram-Schmidt Orthogonalization process. Consider the vectors $\beta_1 = (3, 0, 4), \beta_2 = (-1, 0, 7), \beta_3 = (2, 9, 11)$ in $\mathbb{R}^3$ with standard inner product. Apply Gram-Schmidt Orthogonalization process to $\beta_1, \beta_2, \beta_3$, and obtain an orthonormal basis for $\mathbb{R}^3$.

(2 × 5 = 10 weightage)