C 42787 **(Pages : 3)** **Name.......................................**

Reg. No...................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2023

(CBCSS)

Mathematics

MTH 2C 06—ALGEBRA—II

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

*Answer **all** questions.*
Each question carries 1 weightage.

1. Show that $\dfrac{\mathbb{Z}_3[x]}{<x^2+1>}$ is a field.

2. Prove that $\sqrt{\sqrt[3]{2}-i}$ is algebraic over $\mathbb{Q}$.

3. Show that algebraically closed field F has no proper algebraic extensions.

4. Show that doubling the cube is impossible.

5. Find the splitting field of the polynomial x^3-2 in $Q[x]$.

6. What is the order of $G\left(Q\left(\sqrt[3]{2}, i\sqrt{3}\right)/Q\right)$.

7. Find the order of the Galois group $G(K/Q)$ where K is the splitting field of $x^4-1 \in Q[x]$.

8. Show that the polynomial x^5-2 is solvable by radicals over Q.

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any two questions from each of the following 3 units.
Each question carries 2 weightage.

UNIT I

9. Let $E = F(\alpha)$ be a simple extension of a field F, and let α be algebraic over F. Let the degree of $irr(\alpha, F)$ be $n \geq 1$. Prove that every element β of E can be uniquely expressed in the form $\beta = b_0 + b_1\alpha + ... + b_{n-1}\alpha^{n-1}$, where the b_i are in F.

10. Show that field of complex numbers is an algebraically closed field.

11. Let E be an extension field of F and let $\alpha \in E$ be algebraic of odd degree over F. Show that α^2 is algebraic of odd degree over F, and $F(\alpha) = F(\alpha^2)$.

UNIT II

12. If F is any finite field, then for every positive integer n, show that there is an irreducible polynomial in $F[x]$ of degree n.

13. Let E be a field and let σ be an automorphism of E. Show that $E_\sigma = \{a \in E : \sigma(a) = a\}$ is a subfield of E.

14. Let $\overline{F}$ be algebraic closure of field F and let $E \leq \overline{F}$. Show that if every automorphism of $\overline{F}$ leaving F fixed induces an automorphism of E, then E is the splitting field over F.

UNIT III

15. Let $E = F(s_1, s_2, ... s_n)$ where $s_1, s_2, ... s_n$ are the elementary symmetric functions in $y_1, y_2, ... y_n$. Show that the Galois group of $F(y_1, y_2, ... y_n)$ over E is isomorphic to the symmetric group S_n.

16. Let F be a field of characteristic zero, and let $F \leq E \leq K \leq \overline{F}$, where E is a normal extension of F and K is an extension of F by radicals. Prove that $G(E/F)$ is a solvable group.

17. Let K be a finite extension of degree n of a finite field F of p^r elements. Show that $G(K/F)$ is cyclic of order n, and is generated by σ_{p^r}, where for $\alpha \in K$, $\sigma_{p^r}(\alpha) = \alpha^{p^r}$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries 5 weightage.

18. a) Let F be a field. Prove that an ideal $< p(x) \neq \{0\}$ of $F[x]$ is maximal if and only if $p(x)$ is irreducible over F.

    b) Prove that if F is a field, every proper nontrivial prime ideal of $F[x]$ is maximal.

19. a) Prove that if E is a finite extension field of a field F, and K is a finite extension field of E, then K is a finite extension of F, and $[K:F]=[K:E][E:F]$.

    b) Show that if α and β are conjugate over Q then $Q(\alpha)$ and $Q(\beta)$ are isomorphic fields.

20. State and prove Isomorphism Extension Theorem.

21. Let K be a finite normal extension of a field F,with Galois group $G(K/F)$. For each intermediate field E with $F \leq E \leq K$, let $\lambda(E) = G(K/E)$. Show that :

    a) Fixed field of $G(K/E)$ in K is E ;

    b) λ is one to one on the set of all intermediate fields ; and

    c) If E is a normal extension of F, then $G(K/E)$ is a normal subgroup of $G(K/F)$.

(2 × 5 = 10 weightage)