C 42790 (Pages : 5) Name.......................................

Reg. No...................................

**SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2023**

(CBCSS)

Mathematics

MTH 2C 09—ODE AND CALCULUS OF VARIATIONS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

**Part A**

*Answer **all** questions.*
*Each question carries 1 weightage.*

1. Give confluent hypergeometric equation.

2. Find the first three terms of the Legendre series of

$$f(x) = \begin{cases} 0 & \text{if } \quad -1 \le x < 0, \\ x & \text{if } \quad 0 \le x \le 1. \end{cases}$$

3. Determine whether the following functions if positive definite, negative definite, or neither :

$$-2x^2 + 3xy - y^2.$$

4. Show that $(0, 0)$ is a simple critical point of the system :

$$\begin{cases} \dfrac{dx}{dt} = -2x + 3y + xy \\ \dfrac{dy}{dt} = -x + y - 2xy^2 \end{cases}.$$

5. Describe the phase portrait of the system

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = 0 \end{cases}.$$

**Turn over**

6. Show that $f(x, y) = xy^2$ satisfies a Lipschitz condition on any rectangle $a \le x \le b$ and $c \le y \le d$.

7. Describe Picard's iteration method.

8. Give the solution, if exists, to the initial value problem

$$y' = t,\ y(0) = 1.$$

(8 × 1 = 8 weightage)

## Part B

*Answer any* **two** *questions from each of the following* **three** *units.*
*Each question carries* 2 *weightage.*

### Unit I

9. Express $\sin^{-1} x$ in the form of a power series by solving the equation

$$y' = \left(1 - x^2\right)^{-1/2},\ y(0) = 0$$

in two ways.

10. Locate and classify singular points on the $x$-axis of the differential equation

$$x^2 \left(x^2 - 1\right)^2 y'' - x(1 - x)y' + 2y = 0.$$

11. Find the general solution of the differential equation

$$\left(2x^2 + 2x\right)y'' + (1 + 5x)y' + y = 0$$

near the singular point $x = 0$.

UNIT II

12. If the two solutions

$$\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases} \quad \text{and} \quad \begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$$

of the homogeneous system

$$\begin{cases} \dfrac{dx}{dt} = a_1(t)\,x + b_1(t)\,y \\ \dfrac{dy}{dt} = a_2(t)\,x + b_2(t)\,y \end{cases}$$

are linearly independent on $[a, b]$, then prove that

$$\begin{cases} x = c_1 x_1(t) + c_2 x_2(t) \\ y = c_1 y_1(t) + c_2 y_2(t) \end{cases}$$

is the general solution of the homogeneous system on this interval.

13. Prove that the function $\mathrm{E}(x, y) = ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$, and is negative definite if and only if $a < 0$ and $b^2 - 4ac < 0$.

14. Determine the nature and stability properties of the critical point $(0, 0)$ for the following linear autonomous system :

$$\begin{cases} \dfrac{dx}{dt} = 4x - 3y \\ \dfrac{dy}{dt} = 8x - 6y \end{cases}.$$

Turn over

Unit III

15. Let $y_p(x)$ be a non-trivial solution of Bessel's equation on the positive $x$-axis. If $0 \leq p < 1/2$, then prove that every interval of length $\pi$ contains at least one zero of $y_p(x)$; if $p = 1/2$, then prove that the distance between successive zeros of $y_p(x)$ is exactly $\pi$ ; and if $p > 1/2$, then prove that every interval of length $\pi$ contains at most one zero of $y_p(x)$.

16. Given $\frac{dy}{dx} = x + y$ with the initial condition $y(0) = 1$. By Picard's iteration method, find approximate value of $y$ for $x = 0.2$ and $x = 1$.

17. Prove that the geodesics on a sphere are arcs of great circles.

(6 × 2 = 12 weightage)

**Part C**

*Answer any* **two** *questions.*
*Each question carries* 5 *weightage.*

18. (a) Find a power series solution of the form $\sum a_n x^n$ of the differential equation

$$y' + y = 1.$$

(b) Show that equation

$$4x^2 y'' - 8x^2 y' + \left(4x^2 + 1\right) y = 0$$

has only one Frobenius series solution and find it.

19. (a) Derive Rodrigue's formula for Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n.$$

(b) If the roots $m_1$ and $m_2$ of

$$m^2 - (a_1 + b_2)\,m + (a_1 b_2 - a_2 b_1) = 0$$

are real, distinct, and of the same sign, then prove that the critical point $(0, 0)$ of the system

$$\begin{cases} \dfrac{dx}{dt} = a_1 x + b_1 y \\ \dfrac{dy}{dt} = a_2 x + b_2 y \end{cases}$$

is a node.

20. (a) If $q(x) < 0$, and if $u(x)$ is a nontrivial solution of

$$u'' + q(x)\,u = 0$$

then prove that $u(x)$ has at most one zero.

(b) For the following nonlinear system :

i) Find the critical points ;

ii) Find the differential equation of the paths ; and

iii) Solve this equation to find the paths

$$\begin{cases} \dfrac{dx}{dt} = e^y \\ \dfrac{dy}{dt} = e^y \cos x \end{cases}$$

21. (a) Find the curve joining two points $(x_1, y_1)$ and $(x_2, y_2)$ that yields a surface of revolution of minimum area when revolved about the $x$-axis.

(b) State and prove Picard's theorem.

(2 × 5 = 10 weightage)