D 51308 **(Pages : 3)** **Name.....................................**

Reg. No.................................

THIRD SEMESTER M.Sc. (CBCSS) [REGULAR/SUPPLEMENTARY] DEGREE EXAMINATION, NOVEMBER 2023

Mathematics

MTH 3C 11—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each questions carries a weightage of 1.

1. State the chain rule for multivariable functions.
2. Let D be an open subset of $\mathbb{R}^n$ and $a \in \mathcal{D}$ and let $F : \mathbb{R}^n \to \mathbb{R}^m$. Define the directional derivative of F at **a** in the direction **u**.
3. Prove that any reparametrization of a regular curve is regular.
4. Compute the curvature of the curve $\gamma(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$.
5. Is the surface $x^2 + y^2 + z^4 = 1$ smooth ? Justify your answer.
6. Calculate the first fundamental form of the surface $\sigma(u, v) = (\cos hu, \sin hu, v)$. What kind of surface is this ?
7. Find the equation of the tangent plane of the surface patch $\sigma(r, \theta) = \left(r\cosh\theta, r\sinh\theta, r^2\right)$ at the point (1, 0, 1).
8. Show that the Weingarten map changes sign when the orientation of the surface changes.

(8 × 1 = 8 weightage)

Part B

Answer **six** *questions choosing two from each unit.*
Each question carries a weightage of 2.

Unit I

9. Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, f is differentiable at $x_0 \in E$, g maps an open set containing $f(E)$ into $\mathbb{R}^k$ and g is differentiable at $f(x_0)$. Prove that $F : E \to \mathbb{R}^k$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0)) f'(x_0)$.

10. Prove that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

11. If [A] and [B} are n by n matrices, then show that $\det([B][A]) = \det[B]\det[A]$.

Unit II

12. Prove that a parametrized curve has a unit-speed reparametrization if and only if it is regular.

13. Let γ be a unit-speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Prove that γ is a parametrization of (part of) a circle.

14. Show that $\gamma(t) = \left(\cos^2 t - \frac{1}{2}, \sin t \cos t, \sin t\right)$ is a parametrization of the curve of intersection of the circular cylinder of radius $\frac{1}{2}$ and axis the z-axis with the sphere of radius 1 and centre $\left(-\frac{1}{2}, 0, 0\right)$.

Unit III

15. What is meant by an oriented surface ? Show that *Möbius band* is not orientable.

16. Show that the normal curvature of any curve on a sphere of radius r is $\pm\frac{1}{r}$.

17. Prove that the area of a surface patch is unchanged by reparametrization.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. State and prove the implicit function theorem.

19. Prove the following :

(a) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|A\| < \infty$ and A is a uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$.

(b) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and c is a scalar, then $\|A + B\| \le \|A\| + \|B\|, \|cA\| = |c|\|A\|$.

(c) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then $\|BA\| \le \|B\|\|A\|$.

20. Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Prove that its torsion is given by $\gamma = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$, where the × indicate the vector product and the dot denotes *d/dt*.

21. Let U and $\tilde{U}$ be open subsets of $\mathbb{R}^2$ and let $\sigma : U \to \mathbb{R}^3$ be a regular surface patch. Let $\Phi : \tilde{U} \to U$ be a bijective smooth map with smooth inverse map $\Phi^{-1} : U \to \tilde{U}$. Prove that $\tilde{\sigma} = \sigma \circ \Phi : \tilde{U} \to \mathbb{R}^3$ is a regular surface path.

(2 × 5 = 10 weightage)