D 51310 (Pages : 2) Name..................................

Reg. No..............................

THIRD SEMESTER M.Sc. (CBCSS) [REGULAR/SUPPLEMENTARY] DEGREE EXAMINATION, NOVEMBER 2023

Mathematics

MTH 3C 13—FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each questions carries a weightage of 1.

1. Is norm a linear mapping ? Justify your answer.
2. Find the intersection of the unit ball in C[0, 1] with the subspace span $\{t\}$, where C[0, 1] denotes the set of all continuous functions on [0, 1] equipped with the supremum norm.
3. Show that the inner product $\langle x, y\rangle$ is a continuous function with respect to both the variables.
4. Prove that for any two subspaces L_1 and L_2 of a Hilbert space H, $(L_1 + L_2)^{\perp} = L_1^{\perp} \cap L_2^{\perp}$.
5. State the Hahn Banach Extension theorem.
6. If A is a bounded operator on a normed space, then show that the set ker A = $\{x : Ax = 0\}$ is a closed subspace.
7. State the Banach open map theorem.
8. If A and B are invertible operators prove that AB is also invertible.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions choosing two from each unit.*
Each question carries a weightage of 2.

Unit I

9. If X_0 is a closed subspace of X, then show that the quotient space X/X_0 can be equipped with a norm given by the formula $\|[x]\| = \inf\{\|x - y\|, y \in X_0\}$ for $[x] \in X/X_0$.
10. For every sequence of scalars $a = (a_i)$ and $b = (b_i)$ and for $1 \le p \le \infty$ prove that $\|a+b\|_p \le \|a\|_p + \|b\|_p$.
11. Prove that if $p \ge q \ge 1$, then the sequence space $l_q \subset l_p$.

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Unit II

12. State and prove the Bessel's inequality.
13. Prove that if $\{f_i\}$ is a complete system in a Hilbert space H and $x \perp f_i$, then $x = 0$.
14. Prove that f is a bounded functional if and only if f is a continuous functional.

Unit III

15. Prove that any two norms on a finite dimensional space are equivalent.
16. Show that the shift operator in l_2 defined by $Tx = (0, a_1, a_2, \ldots, a_n \ldots)$ for $a_n \in l_2$ satisfies $\|Tx\| = \|x\|$ for every x and $\|T\| = 1$.
17. Let H be a Hilbert space and $A : H \mapsto H$ be a linear operator. Prove that A is compact if and only if its adjoint A* is compact.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. Prove that the sequence space $l_p, 1 \leq p < \infty$ is a complete normed space.
19. Prove that any two separable infinite dimensional Hilbert spaces H_1 and H_2 are isometrically equivalent.
20. Let M be a closed convex sets in a Hilbert space H. Let $\rho(x, M)$ be the distance of x to the set M. Prove that there exists a unique $y \in M$ such that $\rho(x, M) = \|x - y\|$.
21. Prove that for any normed space X, the dual space X* is complete.

(2 × 5 = 10 weightage)

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