C 42021 **(Pages : 3)** **Name..................................**

Reg. No..............................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2023

(CBCSS)

Mathematics

MTH4C15—ADVANCED FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question carries a weightage 1.

1. Define the Spectrum of a bounded operator and explain its classifications.
2. If X is infinite dimensional, show that the identity operator on X is not compact.
3. Show that the eigen vectors of a self adjoint operator corresponding to distinct eigen values are orthogonal.
4. Let $\varphi(t) \in K[a, b]$. Then show that there exists a sequence of polynomials $P_n(t) \searrow \varphi(t)$ as $n \to \infty$ for all $t \in [a, b]$.
5. Define closed graph operator and give an example.
6. Define Schauder basis and give an example.
7. If X* is a seperable space, then show that X is also seperable.
8. Define Banach Algebra and give an example.

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each module.*
Each question carries a weightage 2.

Module I

9. Let T be a compact operator on X and $\lambda \neq 0$. Then show that Δ_λ is a closed subspace of X.

Turn over

10. Let H be a seperable Hilbert space and T be a non-zero compact self-adjoint operator on H. Show that there exist an orthonormal basis consisting of eigen vectors of T.

11. Find the spectrum of the operator K on $L^2\,[0,1]$ given by $(Kf)(t)=\int_0^1 k(t,s)\,f(s)\,ds$, where

$$k(t,s)=\begin{cases}1, & s\le t\\ 0, & s>t\end{cases}.$$

Module II

12. P be a projection on a Hilbert space H. Show that P is self-adjoint if and only if ker (P) $\perp$ Im (P).

13. Define Strong convergence of sequence of operators. Also show that strong limit of a sequence of orthoprojections is an orthoprojection.

14. Let $P_n(t)$ and $Q_n(t)$ be sequences of polynomials. Assume that for all

$t\in[m,M]$, $Q_n(t)\searrow\psi(t)\in K$ and $P_n(t)\searrow\varphi(t)\in K$. Let $\psi(t)\le\varphi(t)$ for all $t\in[m,M]$. Then show

that $\lim\limits_{n\to\infty} Q_n(A)=:B_1\le B_2:=\lim\limits_{n\to\infty} P_n(A)$.

Module III

15. State and prove the Banach Open Mapping Theorem.

16. Define Sublinear functional and show that the *Minkowski functional* is sublinear.

17. State and prove Gelfand's theorem on maximal ideals.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage 5.

18. (i) Let T be a compact operator on X, $\lambda\ne 0$. Show that $\Delta_\lambda = X \Rightarrow \text{Ker } T_\lambda = 0$.

(ii) Define *Voltera operator* T and show that $1\notin\sigma_p(T)$.

19. State and prove minimax principle.

20. Show that a sequence of operators $T_n \in L(X, Y)$ converges strongly to an operator $T \in L(X, Y)$ if and only if :

(i) The sequence $\{T_n(x)\}$ converges for any x from a dense subset of X,

(ii) There exist $C > 0$ such that $\|T_n\| \leq C$

21. (i) Let $K \subset X$ be a convex set. Then show that K is norm closed if and only if K is ω-closed.

(ii For a real Banach space X, show that the unit ball $\mathcal{D}(X^*) = \{f \in X^* \ \|f\| \leq 1\}$ is a compact set in the ω^*—topology.

(2 × 5 = 10 weightage)