C 42026 **(Pages : 3)** **Name.......................................**

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2023

(CBCSS)

Mathematics

MTH4E08—COMMUTATIVE ALGEBRA

(2019 Admission onwards)

Time : Three Hours Maximum Weightage : 30

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Find all units in the ring $\mathbb{Z}_6$ of integers mod 6.
2. Find the nilradical of the ring $\mathbb{Z}_{12}$ of integers mod 12.
3. Let $\mathbb{Z}_{10}$ be an $\mathbb{Z}$-module under the usual action. Verify whether N = {0, 4, 8} is a submodule of $\mathbb{Z}_{10}$.
4. Let $S^{-1}A$ be a ring of fractions and $f : A \to S^{-1}A$ be the natural inclusion homomorphism. Show that for some $a \in A$ if $f(a) = 0$ then $as = 0$ for some $s \in S$.
5. Let $\mathbb{Z}$ be the ring of integers and $S = \{3^n : n \geq 0\}$. Verify whether 4/6 is in $S^{-1}\mathbb{Z}$.
6. Give a factorizations of 6 into irreducibles in $\mathbb{Z}\left(\sqrt{-5}\right)$ other than $6 = 3 \times 2$.
7. Verify whether $\sqrt{2}$ is integral over $\mathbb{Z}$ in the field $\mathbb{R}$ of reals.
8. Show that the ring $\mathbb{Z}$ of integers doesnot have d.c.c. on ideals.

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each unit.*
Each question has weightage 2.

Unit I

9. Show that every nonzero prime ideal in a principal ideal domain is a maximal ideal.
10. Let M, N be , A-modules and $f : M \to N$ be an A-module homomorphism. Show that *ker f* is a submodule of M and *Im f* is a submodule of N.
11. Show that $0 \to M' \xrightarrow{f} M$ is an exact sequence if and only if f is injective.

Turn over

Unit II

12. Let $\mathbb{Z}$ be the ring of integers and $S = \{p^n : n \geq 0\}$ where p is a prime. Find all units in $S^{-1}\mathbb{Z}$.

13. Let M, N be A-modules and $\phi : M \to N$ be an , A-module homomorphism. Show that if ϕ is injective then $\phi_p : M_p \to N_p$ is injective for all prime ideals P of A.

14. Let q be a p-primary ideal in a ring A. Show that if $x \notin q$ then $r(q : x) = p$.

Unit III

15. Let A be a subring of a ring B such that B is integral over A. Let b be an ideal of B and $a = b \cap A$. Show that B/b is integral over A/a.

16. Let A be a subring of B and $C \subseteq B$ be the integral closure of A in B and S be a multiplicatively closed subset of A. Show that $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.

17. Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of A-modules. Show that if M is Artinian then M' and M'' are Artinian.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. (a) Define local ring.

(b) Verify whether $\mathbb{Z}_8$ is a local ring.

(c) Let A be a ring and m be a maximal ideal of A such that every element of $A - m$ is a unit. Show that A is a local ring.

19. (a) Define flat A-module.

(b) Show that the following are equivalent for an A-module N.

(i) N is flat.

(ii) If $0 \to M' \longrightarrow M \longrightarrow M'' \to 0$ be any exact sequence of A-modules then the tensored sequence.

$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is also exact.

20. (a) Let S be a multiplicatively closed subset of a ring A and q be a p-primary ideal of A. Show that :

    i. if $S \cap p \neq \theta$ then $S^{-1}q = S^{-1}A$.

    ii. if $S \cap p = \theta$ then $S^{-1}q$ is $S^{-1}p$ – primary.

    (b) Let q_1, q_2 be p-primary ideals of a ring A. Show that $q_1 \cap q_2$ is also p-primary.

21. (a) Define Noetherian module.

    (b) Show that if k is a field then the polynomial ring $k[x]$ is Noetherian.

    (c) Let M be an A-module. Prove that M is Noetherian if and only if M is finitely generated.

(2 × 5 = 10 weightage)