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C 42027 (Pages : 4) Name.......................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2023

(CBCSS)

Mathematics

MTH 4E 09—DIFFERENTIAL GEOMETRY

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Sketch level sets of the function $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ at heights 0, and 4.

2. Sketch the following vector field on $\mathbb{R}^2 : X(p) = (p, X(p))$ where $X(x_1, x_2) = (x_2, x_1)$.

3. Define Gauss map. Illustrate with an example.

4. Define geodesic. Show that geodesics have constant speed.

5. Define Levi-Civita parallel vector field. Show that if X and Y are Levi-Civita parallel vector fields along α, then $X \cdot Y$ is constant along α.

6. Find the normal curvature of $-x_1^2 + x_2^2 + x_3^2 = 1$ at a point on the surface in the direction of v.

7. Define parametrized n-surface in $\mathbb{R}^{n+k}, k \geq 0$.

8. Define (i) Differential 1-form ; and (ii) Exact 1-form.

(8 × 1 = 8 weightage)

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Part B

Answer **six** *questions choosing* **two** *from each module.*
Each question has weightage 2.

Module I

9. Find the integral curve through $p=(1,1)$ of the vector field on $\mathbb{R}^2$ given by $X(p)=(p, X(p))$ where $X(p)=(0,1)$.

10. Let $S=f^{-1}(c)$ be an n-surface in $\mathbb{R}^{n+1}$, where $f: U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all $q \in S$, and let X be a smooth vector field on U whose restriction to S is a tangent vector field on S. If $\alpha : I \to U$ is any integral curve of X such that $\alpha(t_0) \in S$ for some $t_0 \in I$, then prove that $\alpha(t) \in S$ for all $t \in I$.

11. Describe the spherical image, when $n=1$ and when $n=2$, of the surface

$$x_1^2 + ... x_{n+1}^2 = 1$$

oriented by $\dfrac{\nabla f}{\|\nabla f\|}$ where f is the function defined by $f(x_1,......, x_n) = x_1^2 + ... + x_{n+1}^2$.

Module II

12. Find the velocity, the acceleration, and the speed of the parametrized curve

$\alpha(t) = (\cos t, \sin t, t)$.

13. Evaluate the Weingarten map L_p for $x_2^2 + x_3^2 = a^2$ in $\mathbb{R}^3$.

14. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p, q \in S$, and let α be a piecewise smooth parametrized curve from p to q. Then prove that parallel transport $P_\alpha : S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.

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Module III

15. Let V be a finite dimensional vector space with dot product and let $L : V \to V$ be a self-adjoint linear transformation on V. Let $S = \{v \in V : v \cdot v = 1\}$ and define $f : S \to \mathbb{R}$ by $f(v) = L(v) \cdot v$. Suppose f is stationary at $v_0 \in S$. Then prove that $L(v_0) = f(v_0)\, v_0$.

16. Find the Gaussian curvature $K : S \to \mathbb{R}$ where S is the hyperboloid

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1.$$

17. State and prove inverse function theorem for n-surface.

(6 × 2 = 12 weightage)

Part C

Answer **two** *questions.*
Each question has weightage 5.

18. (a) Let U be an open subset in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Then prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

(b) Prove that the gradient of f at $p \in f^{-1}(c)$ is orthogonal to $f^{-1}(c)$ at p.

19. Let C be an oriented plane curve. Then prove that there exists a global parametrization of C if and only if C is connected.

20. Let S be an n-surface in $\mathbb{R}^{n+1}$ and let $p \in S$, and let $v \in S_p$. Then prove that there exists an open interval I containing 0 and a geodesic $\alpha : I \to S$ such that :

(i) $\alpha(0) = p$ and $\dot{\alpha}(0) = v$.

(ii) If $\beta : \tilde{I} \to S$ is any other geodesic in S with

$\beta(0) = p$ and $\dot{\beta}(0) = v$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

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21. (a) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let $p \in S$. Let Z be any non-zero normal vector field on S such that $N = \frac{Z}{\| Z \|}$ and let $\{v_1, \ldots\ldots, v_n\}$ be any basis for S_p. Then prove that

$$K(p) = (-1)^n \det \begin{pmatrix} \nabla_{v_1} Z \\ \vdots \\ \nabla_{v_n} Z \\ Z(p) \end{pmatrix} \Bigg/ \| Z(p) \|^n \det \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ Z(p) \end{pmatrix}.$$

(b) Let S_1 be an n-surface in $\mathbb{R}^{n+1}$ and let S_2 be an m-surface in $\mathbb{R}^{m+1}$. Suppose $\varphi : S_1 \to \mathbb{R}^{m+1}$ is a smooth map such that $\varphi(S_1) \subset S_2$. Show that $d\varphi = T(S_1) \to T(S_2)$.

(2 × 5 = 10 weightage)

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