C 42029 **(Pages : 3)** **Name..................................**

Reg. No...............................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2023

(CBCSS)

Mathematics

MTH4E11—GRAPH THEORY

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question carries weightage 1.

1. Illustrate through an example : *vertex cut of a graph*.
2. Prove that a vertex v of a tree G is a cut vertex of G if $d(v) > 1$.
3. If G is a block with $v \geq 3$, then prove that any two edges of G lie on a common cycle.
4. State the *marriage theorem*.
5. Prove that a matching M in G is a maximum matching if G contains no M-augmenting path.
6. Describe k-colouring of a graph and k-chromatic graph.
7. Prove that every k-chromatic graph has at least k vertices of degree at least $k - 1$.
8. Define subdigraph.

(8 × 1 = 8 weightage)

Part B

Answer **six** *questions choosing two from each Module.*
Each question carries weightage 2.

Module I

9. Prove that every connected graph contains a spanning tree.
10. If G is Eulerian, then prove that any trail in G constructed by Fleury's algorithm is an Euler tour of G.

Turn over

11. Let G be a simple graph and let u and v be nonadjacent vertices in G such that

$$d(u) + d(v) \geq v.$$

Then prove that G is Hamiltonian if and only if G + uv is Hamiltonian.

MODULE II

12. Let l be a feasible vertex labelling of G. If G_1 contains a perfect matching M*, then prove that M* is an optimal matching of G.

13. Let M and N be disjoint matchings of G with $|M| > |N|$. Then prove that there are disjoint matchings M′ and N′ of G such that $|M'| = |M| - 1$, $|N'| = |N| - 1$, and $M' \cup N' = M \cup N$.

14. Prove that $\alpha + \beta = \nu$.

MODULE III

15. If G is a plane graph, then prove that :

$$\sum_{f \in F} d(f) = 2\varepsilon.$$

16. If a bridge B has three vertices of attachment v_1, v_2 and v_3, then prove that there exists a vertex v_0 in V (B)\V (C) and three paths P_1, P_2 and P_3 in B joining v_0 to v_1, v_2 and v_3, respectively, such that, for $i \neq j$. P_i and P_j have only the vertex v_0 in common.

17. If G is non-planar, then prove that at least one of H_1 and H_2 is also non-coplanar.

(6 × 2 = 12 weightage)

Part C

Answer **two** *questions.*
Each question carries weightage 5.

18. (a) Let T be a spanning tree of a connected graph G, and let e be any edge T. Then prove that :

    (i) The cotree $\overline{T}$ contains no bond of G.

    (ii) $\overline{T} + e$ contains a unique bond of G.

    (b) Show that G is loopless and has exactly one spanning tree T, then G = T.

19. (a) Prove that a connected graph has an Euler trail if and only if it has at most two vertices of odd degree.

(b) If G is a simple graph with $v \geq 3$ and $\delta \geq v/2$. then prove that G is hamiltonian.

20. (a) Prove that a matching M in G is a maximum matching if and only if G contains no M-augmenting path.

(b) Let M be a matching and K be a covering such that $|M| = |K|$. Then prove that M is a maximum matching and K is a minimum covering.

21. (a) If G is a connected simple graph and is neither an odd cycle nor a complete graph, then prove that $\chi \leq \Delta$.

(b) For any graph G, prove that $\pi_k(G)$ is a polynomial in k of degree v, with integer co-efficients, leading term k^v and constant term zero. Also prove that the coefficients of $\pi_k(G)$ alternate in sign.

(2 × 5 = 10 weightage)